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École Doctorale de mathématiques de la région Paris-Sud
École Doctorale des sciences et de technologies

Discipline : Mathématiques

Thèse de doctorat

Présentée et soutenue par:

Kamel ATTAR

Le 16 Juin 2015
à l'école doctorale de mathématiques de la région Paris-Sud

LE MODÈLE DE GINZBURG-LANDAU AVEC CHAMP MAGNÉTIQUE VARIABLE

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À mes parents

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LE MODÈLE DE GINZBURG-LANDAU AVEC CHAMP MAGNÉTIQUE VARIABLE

Résumé

La thèse de doctorat comporte trois parties rédigées en anglais. Les deux premières parties correspondent principalement à l'étude de l'énergie de l'état fondamental. La dernière partie est consacrée à l'analyse de l'effet de '*pinning*' dans la supraconductivité.

Dans une première partie de cette thèse, nous considérons la fonctionnelle de Ginzburg -Landau avec un champ magnétique variable appliqué dans un domaine borné et régulier de dimension 2. Nous déterminons le comportement asymptotique du paramètre d'ordre dans le régime où le paramètre de Ginzburg-Landau et le champ magnétique sont grands et de même ordre. Comme conséquence, nous montrons que le paramètre d'ordre est localisé asymptotiquement dans la region où le profil du champ magnétique appliqué est petit.

Dans une autre partie, nous considérons la fonctionnelle de Ginzburg -Landau avec un champ magnétique variable appliqué dans un domaine borné et régulier de dimension 2. Le profil du champ magnétique appliqué varie régulièrement et peut s'annuler exactement à l'ordre 1 le long d'une courbe. En supposant que la l'intensité du champ magnétique appliqué varie entre deux échelles caractéristiques, et que le paramètre de Ginzburg- Landau tend vers l'infini, nous déterminons une formule asymptotique précise pour minimiser l'énergie et montrer que les minimiseurs de l'énergie ont des vortex. Nous mettons en évidence que la présence d'un champ magnétique variable implique que la distribution de la vorticité dans l'échantillon n'est pas uniforme.

Dans la dernière partie, nous étudions l'énergie de Ginzburg-Landau d'un supraconducteur avec un champ magnétique variable et un terme de "pinning" dans un domaine borné et régulier de dimension 2. En supposant que le paramètre de Ginzburg-Landau et l'intensité du champ magnétique sont grands et de même ordre, nous déterminons une formule asymptotique précise pour l'énergie. De plus, nous discutons l'existence des solutions non-triviales et déterminons le comportement asymptotique du troisième champ critique de la supraconductivité.

Mots clés

Opérateur de Schrödinger magnétique, théorie spectrale, analyse semi-classique, fonctionnelle de Ginzburg-Landau avec champ magnétique variable.

THE GINZBURG-LANDAU MODEL WITH A VARIABLE MAGNETIC FIELD

Abstract

The PHD thesis (thèse de doctorat) has three parts, the first and the second part correpond mainly to study the groundstate energy, the last one being devoted to the analysis of the '*pinning*' effect in superconductivity.

In a first part of this thesis, we consider the Ginzburg-Landau functional with a variable applied magnetic field in a bounded and smooth two dimensional domain. We determine an accurate asymptotic formula for the minimizing energy when the Ginzburg-Landau parameter and the magnetic field are large and of the same order. As a consequence, it is shown how bulk superconductivity decreases in average as the applied magnetic field increases.

In another part, we consider the Ginzburg-Landau functional with a variable applied magnetic field in a bounded and smooth two dimensional domain. The profile of the applied magnetic field varies smoothly and is allowed to vanish non-degenerately along a curve. Assuming that the strength of the applied magnetic field varies between two characteristic scales, and that the Ginzburg-Landau parameter tends to ∞, we determine an accurate asymptotic formula for the minimizing energy and show that the energy minimizers have vortices. The new aspect in the presence of variable magnetic field is that the distribution of vortices in the sample is not uniform.

In the final part, we study the Ginzburg-Landau energy of a superconductor with a variable magnetic field and a pinning term in a bounded and smooth two dimensional domain Ω. Supposing that the Ginzburg-Landau parameter and the intensity of magnetic field are large and of the same order, we determine an accurate asymptotic formula for the minimizing energy. Also, we discuss the existence of non-trivial solutions and prove an asymptotics of the third critical field.

Keywords

Magnetic Schrödinger operator, spectral theory, semi-classical analysis, Ginzburg-Landau functional with variable magnetic field, pinning effect, superconducting.

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Chapitre 1

Introduction Générale

1.1 Motivations et notations

1.1.1 Supraconducteurs et champs magnétiques

Dans les années 50, les physiciens Vitali Lazarevitch Ginzburg et Lev Davidovitch Landau ont proposé la théorie éponyme de la supraconductivité.

La supraconductivité est un phénomène caractérisé par l'absence de résistance électrique et l'expulsion du champ magnétique. Elle permettrait de transporter de l'électricité sans perte d'énergie.
A très basses températures, le matériau devient supraconducteur. Lorsqu'il est soumis à un champ magnétique de faible intensité, le champ magnétique est repoussé, comme indique la figure ci-dessous.

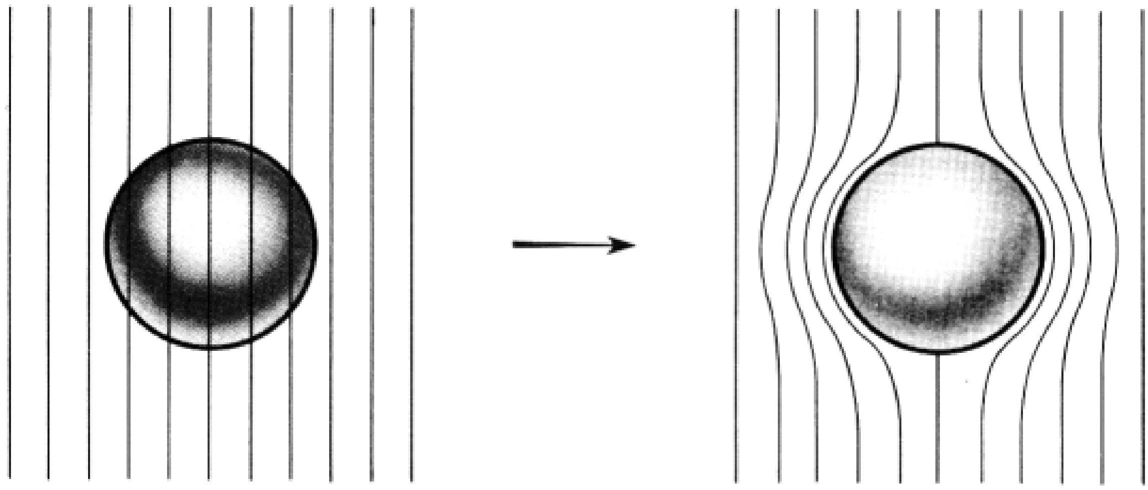

FIGURE 1.1 – Champ magnétique repoussé par un superconducteur.

Dans un supraconducteur, deux effets différents permettent de faire léviter un aimant : l'effet Meissner, et le piégeage des vortex. L'effet Meissner est la propriété de repousser un champ magnétique appliqué en créant un contre champ opposé de même intensité (c.f. Figure 1.5), alors que le piégeage des vortex va maintenir l'aimant à l'endroit où il se trouvait quand le supraconducteur a été refroidi. Autrement dit, l'un repousse, et l'autre piège (sans attirer pour autant). On observe l'un ou l'autre de ces effets selon la force de l'aimant.

1.1.2 Présentation du modèle

L'analyse des problèmes mathématiques liés à la théorie de superconductivité a été intensivement développée depuis une quinzaine d'années. Après une série de réductions, le modèle 2D de Ginzburg-Landau permet de décrire l'état de l'échantillon supraconducteur soumis à un champ magnétique variable B_0 au-dessous de la température critique T_c. Plus précisément, on considère la fonctionnelle

$$\mathcal{E}_{\kappa,H,B_0}(\psi,\mathbf{A}) = \int_\Omega \left[|(\nabla - i\kappa H\mathbf{A})\psi|^2 + \frac{\kappa^2}{2}(1-|\psi|^2)^2 \right] dx + \kappa^2 H^2 \int_{\mathbb{R}^2} |\operatorname{rot}\mathbf{A} - B_0|^2\, dx\,. \quad (1.1.1)$$

Cette fonctionelle a suscité de nombreux travaux mathématiques, on peut citer, les travaux de Helffer, Serfaty, Rubinstein, Fournais, Kachmar, Phillips, X.B. Pan, Sandier, H.Kwek...
Dans cette expression, $\Omega \subset \mathbb{R}^2$ est un ensemble ouvert, qui dans notre étude est toujours supposé *simplement connexe*, borné et à bord régulier. On peut imaginer qu'il correspond par exemple à la section horizontale d'un cylindre vertical infiniment long.
Il sera pratique de soustraire la constante $\frac{\kappa^2}{2}|\Omega|$ de la fonctionnelle $\mathcal{E}_{\kappa,H,B_0}$. Ceci change juste l'énergie du point zéro et n'a aucune conséquence physique.
La fonction B_0 (champ magnétique variable) est initialement définie sur $\mathbb{R}^2$. Lorsque le domaine Ω est simplement connexe, on peut remplacer le domaine d'intégration de $|\operatorname{rot}\mathbf{A} - B_0|^2$ sur $\mathbb{R}^2$ par Ω (cf. [14, Section 11.5]). Ainsi, nous considérons :

$$\widehat{\mathcal{E}}_{\kappa,H,B_0}(\psi,\mathbf{A}) = \int_\Omega \left[|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right] dx + \kappa^2 H^2 \int_\Omega |\operatorname{rot}\mathbf{A} - B_0|^2\, dx\,. \quad (1.1.2)$$

1.1.3 Notations

Pour la paire $(\psi,\mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1(\Omega;\mathbb{R}^2)$ de l'énergie $\widehat{\mathcal{E}}_{\kappa,H,B_0}$:

- La fonction $\psi : \Omega \longrightarrow \mathbb{C}$ est appelée le *'paramètre d'ordre'* en physique, sorte de *'pseudo-fonction d'onde'*, qui décrit l'état local du matériau. Dans la théorie quantique (microscopique) BCS de Bardeen, Cooper et Schrieffer, $|\psi|^2$ mesure la densité locale de paires de Cooper d'électrons supraconducteurs. Dans la théorie phénoménologique :

  1. Si $|\psi| \neq 0$, alors le matériel est dans la phase purement supraconductrice.
  2. Si $|\psi| = 0$, alors le matériel est dans la phase normale.

- Le champ magnétique induit dans le matériau est rot $\mathbf{A}$ (défini par $\operatorname{rot}\mathbf{A} := \partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1$), $\mathbf{A} : \Omega \longrightarrow \mathbb{R}^2$ étant le *'potentiel-vecteur du champ magnétique'*, qui est donc une fonction à valeurs réelles.

- $B_0 \in C^\infty(\overline{\Omega})$ est le profil du champ magnétique appliqué, variable et satisfait :

$$\begin{cases} |B_0| + |\nabla B_0| > 0 & \text{in } \overline{\Omega} \\ \nabla B_0 \times \vec{n} \neq 0 & \text{on } \Gamma \cap \partial\Omega\,, \end{cases} \tag{1.1.3}$$

où,

$$\Gamma = \{x \in \overline{\Omega} :\ B_0(x) = 0\}\,.$$

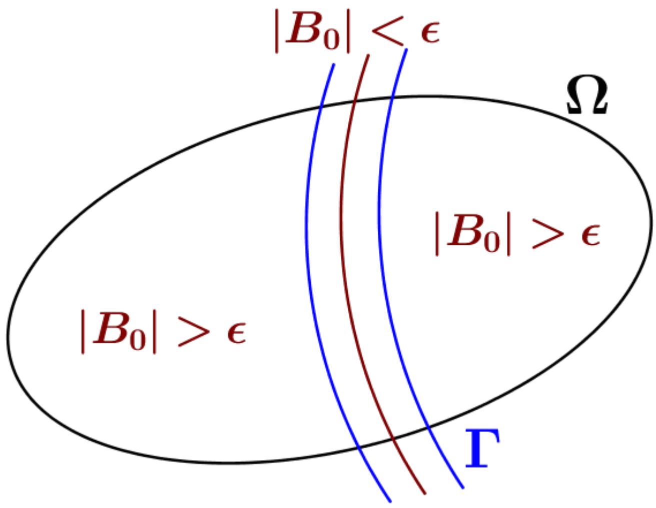

Figure 1.2 – Représentation schématique d'un champ magnétique qui peut s'annuler sur des courbes régulières.

Dans cette fonctionnelle deux paramètres réels interviennent :

- Le paramètre $\kappa > 0$ est une caractéristique du matériau et il est appelé le *'paramètre de Ginzburg-Landau'*. Le matériau est dit du type I si κ est suffisamment petit $\left(\kappa < \frac{1}{\sqrt{2}}\right)$ et il est dit du type II lorsque κ est grand $\left(\kappa > \frac{1}{\sqrt{2}}\right)$. Mathématiquement, cela conduit à l'analyse des divers régimes asymptotiques $\kappa \longrightarrow 0$ ou $\kappa \longrightarrow +\infty$. C'est ce dernier cas qui sera analysé dans notre thèse. Il est aussi appelé limite de London.

- Le paramètre $H > 0$ est l'intensité par rapport du champ magnétique variable appliqué au champ magnétique de référence B_0.

L'hypothèse dans (1.1.3) implique que pour tout ensemble ouvert ω relativement compact dans Ω l'ensemble $\Gamma \cap \omega$ est vide ou une union finie disjointe de courbes régulières. Ici, la définition de la fonctionnelle (1.1.2) est prise comme dans [14]. Dans [13], la mise à l'échelle de l'intensité du champ magnétique externe (notée h) est différente.

1.1.4 Invariance de jauge et équations

La fonctionnelle $\widehat{\mathcal{E}}_{\kappa,H,B_0}$ est invariante par changement de jauge :

$$\begin{cases} \psi \to \psi e^{i\,\Phi} \\ \mathbf{A} \to \mathbf{A} + \nabla\Phi \end{cases}$$

ce qui signifie que pour n'importe quelle fonction $\Phi \in H^2(\Omega, \mathbb{R})$, on a

$$\widehat{\mathcal{E}}_{\kappa,H,B_0}(\psi, \mathbf{A}) = \widehat{\mathcal{E}}_{\kappa,H,B_0}(\psi e^{i\,\Phi}, \mathbf{A} + \nabla\Phi)\,.$$

Deux configurations équivalentes de jauge étant physiquement équivalentes, ceci nous conduit à ne chercher la solution du problème qu'à changement de jauge près. Il est alors mieux de restreindre la fonctionnelle à un sous-espace plus petit $H^1(\Omega, \mathbb{C}) \times H^1_{\rm div}(\Omega)$, avec

$$H^1_{\rm div}(\Omega) = \{\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2) \in H^1(\Omega)^2 \ : \ {\rm div}\,\mathbf{A} = 0 \text{ in } \Omega\,,\ \mathbf{A}\cdot\nu = 0 \text{ on } \partial\Omega\,\}\,,$$

où ν est le vecteur normal extérieur de $\partial\Omega$. En général, nous considérerons la fonctionnelle $\widehat{\mathcal{E}}_{\kappa,H,B_0}$ sur cet espace.

Nous définissons l'énergie de l'état fondamental de Ginzburg-Landau comme l'infimum de la fonctionnelle $\widehat{\mathcal{E}}_{\kappa,H,B_0}$:

$$\widehat{\mathrm{E_g}}(\kappa, H) = \inf\left\{\widehat{\mathcal{E}}_{\kappa,H,B_0}(\psi, \mathbf{A}) \ : \ (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\rm div}(\Omega)\right\}. \tag{1.1.4}$$

Typiquement, Ω correspond au matériau supraconducteur[1], c'est-à-dire au-dessous de sa température critique, entouré par un matériau normal c'est-à-dire au-dessus de sa température critique.
Puisque Ω est bornée, l'existence de minimiseurs est assez classique. Ainsi l'infimum est effectivement un minimum. La démonstration de l'existence de minimiseurs est rappelée par exemple dans le livre de Fournais et Helffer [14, Section 11.2].

Soit $\mathbf{F} : \Omega \to \mathbb{R}^2$ l'unique champ de vecteurs tel que,

$${\rm div}\,\mathbf{F} = 0 \ \text{ et } \ {\rm curl}\,\mathbf{F} = B_0 \text{ dans } \Omega\,, \quad \nu\cdot\mathbf{F} = 0 \text{ sur } \partial\Omega. \tag{1.1.5}$$

Si $(\psi, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H^1_{\rm div}(\Omega)$ est un point critique[2] de $\mathcal{E}_{\kappa,H,B_0}$, donc, $(\psi, \mathbf{A})$ est une solution faible du système constitué des équations suivantes, que nous appelons *'équations de Ginzburg-*

[1]Mentionnons ici que le paramètre de Ginzburg-Landau κ ne dépend que du matériau supraconducteur dans Ω

[2]Nous disons que $(\psi, \mathbf{A})$ est un point critique de $\mathcal{E}_{\kappa,H,B_0}$ si pour chaque $(\widetilde{\psi}, \widetilde{\mathbf{A}})$ a support compact, nous avons :

$$\frac{d}{dt}\mathcal{E}_{\kappa,H,B_0}(\psi + t\widetilde{\psi}, \mathbf{A} + t\widetilde{\mathbf{A}})_{|t=0} = 0.$$

Landau' (c.f. [44, Section 3.2]),

$$\begin{cases} -(\nabla - i\kappa H\mathbf{A})^2\psi = \kappa^2(1-|\psi|^2)\psi & \text{dans} \quad \Omega \\ -\nabla^{\perp}\operatorname{rot}(\mathbf{A}-\mathbf{F}) = \dfrac{1}{\kappa H}\operatorname{Im}(\overline{\psi}\,(\nabla - i\kappa H\mathbf{A})\psi) & \text{dans} \quad \Omega \\ \nu\cdot(\nabla - i\kappa H\mathbf{A})\psi = 0 & \text{sur} \quad \partial\Omega \\ \operatorname{rot}\mathbf{A} = \operatorname{rot}\mathbf{F} & \text{sur} \quad \partial\Omega\,. \end{cases} \tag{1.1.6}$$

Dans les équations ci-dessus, le champ vecteur $\nabla^{\perp}\operatorname{rot}\mathbf{A}$ est défini par

$$\nabla^{\perp}\operatorname{rot}\mathbf{A} = (-\partial_{x_2}(\operatorname{rot}\mathbf{A}), \partial_{x_1}(\operatorname{rot}\mathbf{A})).$$

Le vecteur ν désigne la normale unitaire en un point de $\partial\Omega$ pointant vers l'extérieur.
L'analyse du système (1.1.6), peut être effectuée par des techniques d'équations aux dérivées partielles.
Nous rappelons que ce système est non linéaire. $H^1(\Omega)$ est, (quand Ω est borné et régulier dans $\mathbb{R}^2$) d'injection compacte dans $L^p(\Omega)$ pour tout $p \in [1, +\infty)$. De plus, on peut montrer que la solution dans $H^1(\Omega, \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ du système elliptique (1.1.6) est en fait, quand Ω est régulière, dans $C^\infty(\overline{\Omega}, \mathbb{C}) \times C^\infty(\overline{\Omega}, \mathbb{R}^2)$. (c.f. [14, Théorème F.2.1]).

1.1.5 Les vortex

Quand l'intensité du champs magnétique H augmente progressivement tout en passant par des valeurs critiques (voir Section 1.1.6), le champ magnétique B_0 ne sera plus repoussé. Autrement dit, il traverse l'échantillon de part en part par le biais des vortex qui sont représentés par des disques D qui peuvent être vus comme l'ensemble $\{|\psi|^2 \approx 0\}$.

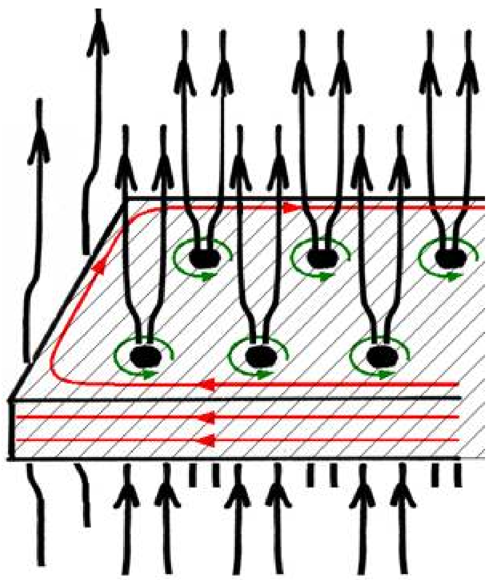

FIGURE 1.3 – Représentation schématique d'un supraconducteur avec tourbillons (vortex).

Pour permettre à ce champ magnétique (représenté en noir) de passer à travers le vortex, le matériau développe des courants supraconducteurs (représentés en vert) :

$$j = \frac{i}{2\kappa H}\left(\psi\overline{(\nabla - i\kappa H\mathbf{A})\psi} - \overline{\psi}(\nabla - i\kappa H\mathbf{A})\psi\right)$$

qui circulent autour de ces disques, sous forme de tourbillon qui justifie le nom 'vortex'. Autour de ces disques, la phase φ de ψ a un nombre d'enroulement non nul $d = \frac{1}{2\pi}\int_{\partial D}\frac{\partial\varphi}{\partial\tau}$ d'éfinie sur le bord du disque, appelé *le degré du vortex*, où les vortex de degré 1 et -1 sont représentés.

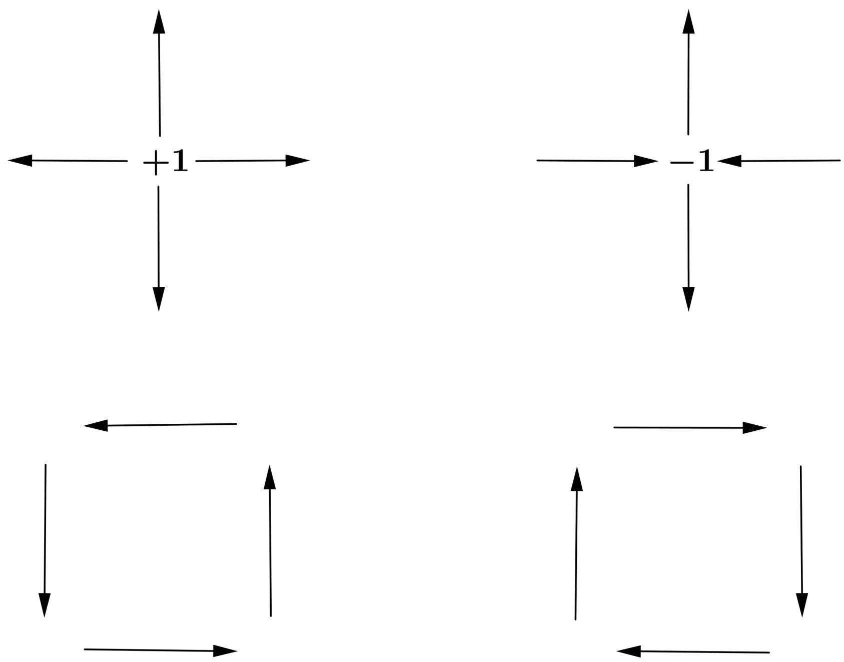

FIGURE 1.4 – Vortex de degré $+1$ et -1, les flèches représentent le courant.

1.1.6 Les champs critiques

Pour un κ donné, le comportement des minimiseurs et plus généralement des points critiques de la fonctionnelle est déterminé en fonction de la valeur du champ magnétique H.

Quand le champ magnétique est constant, il y a trois valeurs principales du H ou champs critiques H_{C_1}, H_{C_2} et H_{C_3}, pour lesquelles des transitions de phase se produisent.

En dessous du premier champ critique H_{C_1}, le supraconducteur est partout dans sa phase supraconductrice $|\psi| \neq 0$ et le champ magnétique ne pénètre pas (cela s'appelle l'effet Meissner ou état de Meissner (c.f. Figure 1.5). Mathématiquement, le champ magnétique induit rot $\mathbf{A}$ est asymptotiquement très petite.

À H_{C_1}, un premier vortex apparaît. Entre H_{C_1} et H_{C_2} les phases supraconductrices et normales (sous la forme de tourbillons) coexistent dans l'échantillon, et le champ magnétique pénètre à travers les vortex. C'est ce qu'on appelle l'état mixte (c.f. Figure 1.6).

À H_{C_2}, quand κ est grand, une deuxième transition de phase se produit. $|\psi| \approx 0$ à l'intérieur de l'échantillon, c'est-à-dire que la supraconductivité dans la plus grande partie de l'échantillon est perdue. Mathématiquement, le champ magnétique induit rot $\mathbf{A}$ et le profil du champ magnétique appliqué B_0 sont asymptotiquement égaux.

Entre H_{C_2} et H_{C_3}, la supraconductivité persiste près de la frontière, c'est ce qu'on appelle la supraconductivité de surface, et après H_{C_3}, la supraconductivité est complètement détruite et $|\psi| = 0$ dans tout l'échantillon. L'échantillon est alors complètement en phase normale.

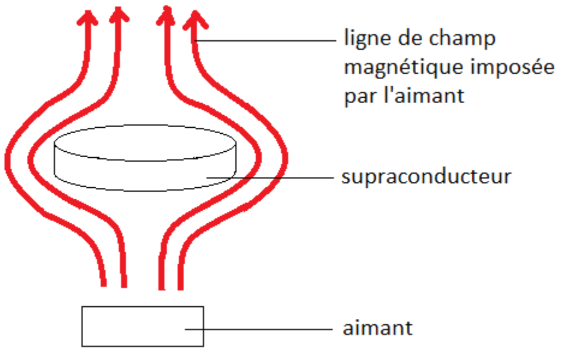

FIGURE 1.5 – Etat Meissner.

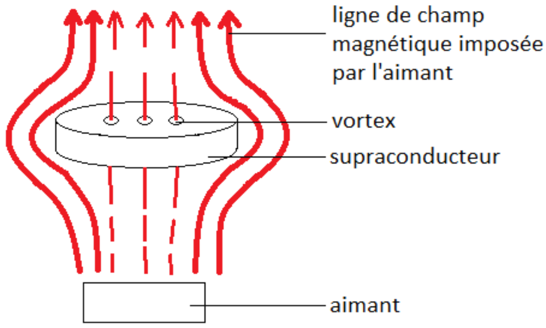

FIGURE 1.6 – Etat mixte.

Nous allons donner maintenant le comportement asymptotique des trois champs critiques dans des plusieurs cas :

Le cas du champ magnétique constant :

$$H_{C_2} = \frac{\kappa}{B_0} \qquad \text{et} \qquad H_{C_3} \sim \frac{\kappa}{\Theta_0\, B_0}\,.$$

Ici, Θ_0 est une constante universelle telle que, $\Theta_0 \in (1/2, 1)$.

Le cas du champ magnétique variable qui ne s'annule pas :

$$H_{C_2} = \frac{\kappa}{\inf\limits_{x\in\overline{\Omega}} B_0} \qquad \text{et} \qquad H_{C_3} \sim \frac{\kappa}{\Theta_0 \inf\limits_{x\in\partial\Omega} B_0} \qquad \text{avec} \left(\inf_{x\in\overline{\Omega}} B_0 > \Theta_0 \inf_{x\in\partial\Omega} B_0\right)\,.$$

Dans le cas où $\left(\inf_{x\in\overline{\Omega}} B_0 < \Theta_0 \inf_{x\in\partial\Omega} B_0\right)$, le phénomène de la supraconductivité de surface disparaît. Ceci nous donne que les comportements asymptotiques de H_{C_2} et H_{C_3} sont égaux.

$$H_{C_2} = H_{C_3} \sim \frac{\kappa}{\Theta_0 \inf\limits_{x\in\partial\Omega} B_0} \,.$$

Le cas du champ magnétique variable qui s'annule le long d'une courbe simple et régulière Γ :

$$H_{C_2} = \frac{\kappa^2}{\lambda_0^{\frac{3}{2}} \min\limits_{x\in\Omega\cap\Gamma} |\nabla B_0(x)|} \qquad \text{et} \qquad H_{C_3} \sim \frac{\kappa^2}{\min\limits_{x\in\partial\Omega\cap\Gamma} \lambda_0(\mathbb{R}_+,\theta(x))^{\frac{3}{2}} |\nabla B_0(x)|}$$

avec

$$\lambda_0^{\frac{3}{2}} \min_{x\in\Omega\cap\Gamma} |\nabla B_0(x)| > \min_{x\in\partial\Omega\cap\Gamma} \lambda_0(\mathbb{R}_+,\theta(x))^{\frac{3}{2}} |\nabla B_0(x)| \,.$$

Ici, λ_0 est introduite dans (4.1.31), $\lambda_0(\mathbb{R}_+,\theta(x))$ est le bas du spectre de l'opérateur qui est défini dans (4.1.33) et $\theta(x)$ désigne l'angle entre $|\nabla B_0(x)|$ et le vecteur normal $-\nu(x)$.
Dans le cas où $\lambda_0^{\frac{3}{2}} \min\limits_{x\in\Omega\cap\Gamma} |\nabla B_0(x)| < \min\limits_{x\in\partial\Omega\cap\Gamma} \lambda_0(\mathbb{R}_+,\theta(x))^{\frac{3}{2}} |\nabla B_0(x)|$, la supraconductivité de surface disparaît. Alors, nous ne distinguons pas entre H_{C_2} et H_{C_3}, de plus

$$H_{C_2} = H_{C_3} \sim \frac{\kappa^2}{\min\limits_{x\in\partial\Omega\cap\Gamma} \lambda_0(\mathbb{R}_+,\theta(x))^{\frac{3}{2}} |\nabla B_0(x)|} \,.$$

Pour ce qui concerne le premier champ critique H_{C_1}, nous ronvoyons à Sandier et Serfaty [44].

1.2 L'énergie de l'état fondamental de la fonctionnelle de Ginzburg-Landau avec un champ magnétique appliqué variable dans un domaine de $\mathbb{R}^2$

1.2.1 Problème posé

Ici on prend κ grand ce qui correspond à une hypothèse que le matériau est fortement de type II. Les propriétés supraconductrices sont décrites par les minimiseurs $(\psi, \mathbf{A})$ de la fonctionnelle $\widehat{\mathcal{E}}_{\kappa,H,B_0}$ définie en (1.1.2). On s'intéresse juste à déterminer le comportement asymptotique du paramètre d'ordre ψ dans le régime où le paramètre de Ginzburg-Landau κ et le champ magnétique H sont grands et de même ordre, i.e.

$$\exists\, \kappa_0 \geq 0\,, \qquad \forall \kappa \geq \kappa_0, \qquad \Lambda_{\min}\, \kappa \leq H \leq \Lambda_{\max}\, \kappa\,, \tag{1.2.1}$$

où $\Lambda_{\min}$ et $\Lambda_{\max}$ sont des constantes strictement positives telle que $\Lambda_{\min} \leq \Lambda_{\max}$.
Comme conséquence, nous montrons que le paramètre d'ordre ψ est localisé asymptotiquement dans la region où $B_0 < \frac{\kappa}{H}$ lorsque H vérifie (1.2.1).

1.2.2 Résultats principaux

Étant donné $b \geq 0$. Nous définissons l'énergie de Ginzburg-Landau dans un ensemble ouvert $\mathcal{D} \subset \mathbb{R}^2$,

$$G_{b,\mathcal{D}}(u) = \int_{\mathcal{D}} \left(b|(\nabla - i\mathbf{A}_0)u|^2 + \frac{1}{2}(1 - |u|^2)^2 \right) dx\,, \qquad \forall u \in H_0^1(\mathcal{D})\,.$$

Ici,

$$\mathbf{A}_0(x) = \frac{1}{2}(-x_2, x_1)\,, \qquad \forall x = (x_1, x_2) \in \mathbb{R}^2\,.$$

Compte tenu de la définition de l'énergie de l'état fondamental $\widehat{\mathrm{E}_\mathrm{g}}$ dans (1.1.4). Nous donnons une estimation asymptotique de $\widehat{\mathrm{E}_\mathrm{g}}$, qui est valable quand H satisfait (1.2.1). Le comportement de l'énergie $\widehat{\mathrm{E}_\mathrm{g}}(\kappa, H)$ implique une fonction auxiliaire $\hat{f}$ qui est la limite de minimum de l'énergie de Ginzburg-Landau G_{b,Q_R} avec $Q_R =]-R/2, R/2[\times]-R/2, R/2[$,

$$\hat{f}(b) = \lim_{R \to +\infty} \frac{\inf\limits_{u \in H_0^1(Q_R)} G_{b,Q_R}(u)}{R^2}\,. \tag{1.2.2}$$

La fonction $\hat{f}$ a les propriétés suivantes :

- Pour tous $b \geq 1$, $\hat{f}(b) = \frac{1}{2}$, et $\hat{f}(0) = 0$.

- $\hat{f} : [0,1] \longrightarrow [0, \frac{1}{2}]$ est une fonction croissante, continue et concave.

-

$$\hat{f}(b) = \frac{b}{2} \ln \frac{1}{b} \left(1 + o(1)\right), \qquad \text{lorsque } b \longrightarrow 0\,. \tag{1.2.3}$$

La fonction $\hat{f}$ a été introduite par Sandier-Serfaty dans [45], puis analysée en [3, 18]. Cette fonction joue un rôle important dans la description de la distribution de la supraconductivité dans la plupart des échantillons à deux et trois dimensions, voir [45], [3, 17, 18] et les articles récents [5, 27].

Théorème 1.2.1. *Supposons que B_0 vérifie* (1.1.3) *et H vérifie* (1.2.1). *Alors, l'énergie de l'état fondamental $\widehat{\mathrm{E}_{\mathrm{g}}}$ satisfait,*

$$\widehat{\mathrm{E}_{\mathrm{g}}}(\kappa, H) = \kappa^2 \int_\Omega \left\{ \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) - \frac{1}{2} \right\} dx + o(\kappa^2)\,, \qquad (\kappa \to +\infty)\,. \tag{1.2.4}$$

Remarque 1.2.2. *Plus précisément, quand κ est grand, on peut montrer l'existence de $C > 0$ et $\tau_0 \in (1,2)$ tels que ;*

$$\left| \widehat{\mathrm{E}_{\mathrm{g}}}(\kappa, H) - \kappa^2 \int_\Omega \left\{ \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) - \frac{1}{2} \right\} dx \right| \leq C\kappa^{\tau_0}\,. \tag{1.2.5}$$

Notons que, le deuxième terme à droite dans (1.2.4) est petit par rapport au premier terme $\kappa^2 \int_\Omega \left\{ \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) - \frac{1}{2} \right\} dx$ qui est donc le terme dominant.
Le théorème 1.2.1 a été prouvé par E. Sandier et S. Serfaty (cf. [45, Théorème 1.4]) lorsque le champ magnétique B_0 est constant ($B_0 = 1$). Cependant, ces auteurs ne contrôlent pas le reste avec la même précision dans [27]. Pour être plus précis, l'énergie de l'état fondamental dans le cas $B_0 = 1$ qui était $\kappa^2 |\Omega| \left\{ \hat{f}\left(\frac{H}{\kappa}\right) - \frac{1}{2} \right\}$, devient maintenant l'intégrale $\kappa^2 \int_\Omega \left\{ \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) - \frac{1}{2} \right\} dx$. L'approche utilisée dans la preuve du Théorème 1.2.1 est différente de celle du [45] et elle s'inspire plutôt de celle dans [17] qui étudie le même problème lorsque $\Omega \subset \mathbb{R}^3$ et B_0 constant.

L'énoncé du Théorème 1.2.3 nous donne que l'énergie magnétique est petite, comparée à celle du terme dominant qui est d'ordre supérieur à $\mathcal{O}(\kappa^{\tau_0})$.

Théorème 1.2.3 (Estimation de l'énergie magnétique.). *Avec les notations et l'hypothèses du Théorème 1.2.1, il existe deux constantes $C > 0$ et $\kappa_0 > 0$, telles que, si $\kappa \geq \kappa_0$, alors l'énergie magnétique satisfait,*

$$(\kappa H)^2 \int_\Omega |\operatorname{rot} \mathbf{A} - B_0|^2\, dx \leq C\kappa^{\tau_0}\,. \tag{1.2.6}$$

Dans notre démonstration, la valeur de τ_0 dépend des propriétés de B_0 : nous trouvons que $\tau_0 = \frac{7}{4}$ lorsque B_0 ne s'annule pas en $\overline{\Omega}$ et que $\tau_0 = \frac{15}{8}$ dans le cas général.

Nous allons maintenant donner une version locale du Théorème 1.2.1. Tout d'abord, si $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$, nous introduisons la densité de l'énergie,

$$e(\psi, \mathbf{A}) = |(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\,.$$

Nous introduisons aussi l'énergie locale de ψ dans un domaine $\overline{\mathcal{D}} \subset \Omega$:

$$\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{D}) = \int_{\mathcal{D}} e(\psi, \mathbf{A})\, dx\,. \tag{1.2.7}$$

De plus, nous definissons l'énergie locale de Ginzburg-Landau de $(\psi, \mathbf{A})$ dans un domaine $\overline{\mathcal{D}} \subset \Omega$ comme suit,

$$\mathcal{E}(\psi, \mathbf{A}; \mathcal{D}) = \mathcal{E}_0(\psi, \mathbf{A}; \mathcal{D}) + (\kappa H)^2 \int_{\Omega} |\operatorname{rot} \mathbf{A} - B_0|^2\, dx\,. \tag{1.2.8}$$

Théorème 1.2.4 (Estimation de l'énergie locale.)**.** *Supposons que $\mathcal{D} \subset \Omega$ soit un ensemble ouvert. Alors, il existe une constante $\kappa_0 > 0$ telle que, lorsque H vérifie* (1.2.1) *et $(\psi, \mathbf{A})$ un minimiseur de* (1.1.2)*, nous avons*

$$\mathcal{E}(\psi, \mathbf{A}; \mathcal{D}) = \kappa^2 \int_{\mathcal{D}} \left\{ \hat{f}\left(\frac{H}{\kappa} |B_0(x)| \right) - \frac{1}{2} \right\} dx + o(\kappa^2)\,. \tag{1.2.9}$$

Dans le cas où $B_0 = 1$, Sandier et Serfaty (cf. [45]) ont donné un comportement asymptotique du $\mathcal{E}(\psi, \mathbf{A}; \mathcal{D})$ quand $(\psi, \mathbf{A})$ est un minimiseur de (1.1.2).

Théorème 1.2.5 (Sandier-Serfaty)**.** *Il existe une constante $\kappa_0 > 0$ telle que, lorsque H vérifie* (1.2.1)*, nous avons*

$$\mathcal{E}(\psi, \mathbf{A}; \mathcal{D}) = \kappa^2\, |\mathcal{D}| \left\{ \hat{f}\left(\frac{H}{\kappa} \right) - \frac{1}{2} \right\} + o(\kappa^2)\,, \qquad (\kappa \to +\infty)\,. \tag{1.2.10}$$

Le théorème suivant nous donne un comportement asymptotique du paramètre d'ordre ψ, quand $(\psi, \mathbf{A})$ est un minimiseur global.

Théorème 1.2.6 (Concentration du paramètre d'ordre.)**.** *Avec les notations et hypothèses du Théorème 1.2.1. Il existe deux constantes positives C, κ_0 et une constante $\tau_1 \in (-1, 0)$ telles que, si $\kappa \geq \kappa_0$, et $\mathcal{D}$ est un ouvert régulier tel que $\overline{\mathcal{D}} \subset \Omega$, alors,*

1. *Si $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ est une solution de* (1.1.6)*, alors,*

$$\frac{1}{2} \int_{\mathcal{D}} |\psi|^4\, dx \leq - \int_{\mathcal{D}} \left\{ \hat{f}\left(\frac{H}{\kappa} |B_0(x)| \right) - \frac{1}{2} \right\} dx + C\kappa^{\tau_1}\,. \tag{1.2.11}$$

2. *Si $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ est un minimiseur de* (1.1.2)*, alors,*

$$\left| \int_{\mathcal{D}} |\psi|^4\, dx + 2 \int_{\mathcal{D}} \left\{ \hat{f}\left(\frac{H}{\kappa} |B_0(x)| \right) - \frac{1}{2} \right\} dx \right| \leq C\kappa^{\tau_1}\,. \tag{1.2.12}$$

La valeur de τ_1 trouvée dépend des propriétés de B_0 : nous trouvons que $\tau_1 = -\frac{1}{4}$ lorsque B_0 ne s'annule pas en $\overline{\Omega}$ et que $\tau_1 = -\frac{1}{8}$ dans le cas général.
Notons que $\hat{f}(\cdot) \in [0, \frac{1}{2}]$, ce qui implique

$$- \int_{\mathcal{D}} \left\{ 2\hat{f}\left(\frac{H}{\kappa} |B_0(x)| \right) - 1 \right\} dx \geq 0\,.$$

Le théorème 1.2.6 a été prouvé par E. Sandier et S. Serfaty (cf. [45]) lorsque $\Omega \subset \mathbb{R}^2$ et B_0 est constant.

Théorème 1.2.7 (Sandier-Serfaty)**.** *Supposons que $\Omega \subset \mathbb{R}^2$, $B_0 = 1$, H vérifie* (1.2.1) *et que $\mathcal{D}$ est un ouvert régulier tel que $\overline{\mathcal{D}} \subset \Omega$. Alors,*

1. *Si $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ est une solution de* (1.1.6)*, alors,*

$$\frac{1}{2}\int_{\mathcal{D}} |\psi|^4\, dx \leq -\left\{\hat{f}\left(\frac{H}{\kappa}\right) - \frac{1}{2}\right\}|\mathcal{D}| + o(1)\,, \quad (\kappa \to +\infty)\,. \tag{1.2.13}$$

2. *Si $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ est un minimiseur de* (1.1.2)*, alors,*

$$\left|\int_{\mathcal{D}} |\psi|^4\, dx + 2|\mathcal{D}|\left\{\hat{f}\left(\frac{H}{\kappa}\right) - \frac{1}{2}\right\}\right| \leq o(1)\,, \quad (\kappa \to +\infty)\,. \tag{1.2.14}$$

1.2.3 Méthodes de démonstration.

Les techniques appliquées dans la démonstration sont étroitement liées à la définition de la fonction $\hat{f}$ (voir (1.2.2)), qui est la densité d'énergie de l'état fondamental associé à un champ appliqué constant b. Elle devient très grande à l'échelle naturelle du problème, mais reste petite à l'échelle de Ω et des variations de B_0. À l'aide de cette fonction nous avons obtenu le comportement asymptotique de l'énergie dans le régime où H et κ sont grands et de même ordre.

La preuve des résultats ci-dessus consiste à subdiviser le domaine Ω en carrés de côté ℓ en excluant ceux qui rencontrent $|B_0| < \epsilon$, et à approximer l'énergie dans chaque carré en approximant B_0 par un B_0 constant, carré dans lesquel on utilisera les bornes supérieures et inférieurs triviales. Les valeurs de ℓ et ϵ sont également à optimiser pour que l'erreur soit la plus petite possible.

Dans l'estimation du reste, nous nous référons aux estimations a priori des solutions des équations de Ginzburg-Landau (1.1.6) qui jouent un rôle essentiel dans le contrôle des erreurs résultant des diverses approximations.

De plus, nous utilisons la version locale de l'énergie pour démontrer que l'énergie magnétique est dans l'erreur, ce qui nous donne l'asymptotique de l'energie de l'état fondamental de Ginzburg-Landau dans un sous-domaine.

1.2.4 Discussion des résultats principaux

Si $\Gamma \neq \varnothing$ et $H = b\kappa$, $\ b > 0$, alors,

$$\hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) - \frac{1}{2} \neq 0 \qquad \text{en } \mathcal{D} = \left\{x \in \Omega : \frac{H}{\kappa}|B_0(x)| < 1\right\}\,,$$

et

$$|\mathcal{D}| \neq 0\,.$$

Par conséquent, pour κ assez grand, nous obtenons le résultat suivant :

$$\int_{\mathcal{D}} |\psi|^4 \, dx > \epsilon_{\mathcal{D}} \qquad \text{dans } L^2(\Omega)\,,$$

où $\epsilon_{\mathcal{D}}$ est une constante positive dépendant de $\mathcal{D}$.
Ceci nous indique que la supraconductivité est localisée dans la région où $B_0 < \frac{\kappa}{H}$ contrairement au cas lorsque le champ magnétique est constant.
Il y a donc une différence importante entre nos résultats et ceux pour le champ magnétique constant. Quand le champ magnétique est d'intensité constante non nulle, (cf. [14]), il existe une constante universelle $\Theta_0 \in (\frac{1}{2}, 1)$ telles que, si $H = b\kappa$ et $b > \Theta_0^{-1}$, alors $\psi = 0$ dans $\overline{\Omega}$. De plus, dans la même situation, lorsque $H = b\kappa$ and $1 < b < \Theta_0^{-1}$, ψ est asymptotiquement petit partout sauf dans un petit voisinage de $\partial\Omega$ (cf. [48]). Notre résultat est dans le même esprit que dans [47], où les auteurs ont établi que, sous l'hypothèse (1.1.3), pour κ assez grand, quand $H = b\kappa^2$ et $b > b_0$, alors $\psi = 0$ in $\overline{\Omega}$. (b_0 est une constante).

1.3 Énergie et vorticité pour un modèle de Ginzburg-Landau avec un champ magnétique variable

Nous allons déterminer une formule asymptotique précise pour le minimum de l'énergie et montrer que les minimiseurs de l'énergie ont des vortex quand l'intensité du champ magnétique appliqué H varie entre deux échelles caractéristiques, et que le paramètre de Ginzburg-Landau κ tend vers l'infini, autrement dit, quand H satisfait,

$$C_{\min}\kappa^{\frac{1}{3}} \leq H \ll \kappa \qquad \text{lorsque } \kappa \longrightarrow +\infty\,, \tag{1.3.1}$$

où $C_{\min}$ est une constante positive.
De plus, nous allons mettre en évidence que la présence d'un champ magnétique variable implique que la distribution de la vorticité dans l'échantillon n'est pas uniforme. Notons que, dans le cas où B_0 est constant, Sandier et Serfaty ont prouvé que la distribution de la vorticité dans l'échantillon est uniforme.

1.3.1 Énoncé des résultats

Théorème 1.3.1. *Supposons que B_0 vérifie* (1.1.3) *et que H vérifie* (1.3.1)*, alors, l'énergie de l'état fondamental dans* (1.1.4) *admet le comportement asymptotique suivant lorsque κ tend vers $+\infty$:*

$$\widehat{\mathrm{E}_{\mathrm{g}}}(\kappa, H) = \kappa^2 \int_\Omega \left\{ \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) - \frac{1}{2} \right\} dx + o\left(\kappa H \ln \frac{\kappa}{H}\right)\,. \tag{1.3.2}$$

Le deuxième terme à droite dans (1.2.4), qui peut s'écrire plus simplement $o(\kappa H \ln \kappa)$ quand (1.3.1) satisfaite, est d'ordre inférieur à celui du premier terme $\kappa^2 \int_\Omega \left\{ \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) - \frac{1}{2} \right\} dx$ qui est le terme dominant.

Sachant que $\hat{f}$ satisfait (1.2.2), nous pouvons réécrire (1.2.4) comme suit,

$$\widehat{\mathrm{E}_{\mathrm{g}}}(\kappa, H) = \frac{1}{2}\left[\int_{\Omega}\left(\kappa H|B_0(x)|\ln\frac{\kappa}{H|B_0(x)|} - 1\right)dx\right](1 + o(1))\,. \tag{1.3.3}$$

Quand le champ magnétique est constant, (1.3.3) est prouvé par Sandier-Serfaty [46] sous la condition :

$$\frac{\ln\kappa}{\kappa} \ll H \ll \kappa\,. \tag{1.3.4}$$

Théorème 1.3.2. *Supposons que* $B_0 = 1$ *et* H *vérifie* (1.3.4)*, alors, l'énergie de l'état fondamental dans* (1.1.4) *satisfait*

$$\widehat{\mathrm{E}_{\mathrm{g}}}(\kappa, H) = \frac{1}{2}|\Omega|\left(\kappa H\ln\frac{\kappa}{H} - 1\right)(1 + o(1))\,, \qquad (\kappa \to +\infty)\,. \tag{1.3.5}$$

La raison pour laquelle nous n'obtenons pas (1.3.3) sous la condition ci-dessus est probablement technique. La méthode consiste à construire des tests de configurations avec une condition au bord de Dirichlet. Nous ne pouvons pas construire des configurations périodiques comme dans [46], parce que le champ magnétique B_0 est variable.
L'approche utilisée dans la preuve du Théorème 1.2.5 est inspirée de celle dans [31] qui étudie le même problème quand $\Omega \subset \mathbb{R}^3$ et B_0 est constante.

Rappelons que quand H vérifie (1.2.1), nous avons obtenu

$$\widehat{\mathrm{E}_{\mathrm{g}}}(\kappa, H) = \kappa^2\int_{\Omega}\left\{\hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) - \frac{1}{2}\right\}dx + o(\kappa H)\,, \qquad (\kappa \longrightarrow +\infty)\,. \tag{1.3.6}$$

Si nous supposons qu'il existe des constantes positives $C_{\min}$ et $C_{\max}$ avec $H(\kappa)$ satisfaisant

$$C_{\min}\kappa^{\frac{1}{3}} \leq H(\kappa) \leq C_{\max}\kappa\,, \tag{1.3.7}$$

alors (1.3.6) et (1.3.2) deviennent

$$\widehat{\mathrm{E}_{\mathrm{g}}}(\kappa, H) = \kappa^2\int_{\Omega}\left\{\hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) - \frac{1}{2}\right\}dx + o\left(\kappa H\left(\left|\ln\frac{H}{\kappa}\right| + 1\right)\right)\,. \tag{1.3.8}$$

En particulier, l'hypothèse faite sur H couvre la situation considérée dans (1.2.1).
Quand l'ensemble $\Gamma = \{x \in \overline{\Omega}, B_0(x) = 0\}$ est une union finie de courbes régulières et que l'intensité de champ magnétique H satisfait

$$\kappa \ll H \ll \kappa^2\,. \tag{1.3.9}$$

Helffer et Kachmar ont donné dans [27] le comportement asymptotique de l'énergie $\mathrm{E}_{\mathrm{g}}(\kappa, H)$ suivant :

Théorème 1.3.3. *Supposons que* B_0 *vérifie* (1.1.3) *et* H *vérifie* (1.3.9)*, alors,*

$$\widehat{\mathrm{E}_{\mathrm{g}}}(\kappa, H) = \kappa^2\int_{\Omega}\hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)dx + \frac{\kappa^3}{H}o(1)\,, \qquad (\kappa \to +\infty)\,. \tag{1.3.10}$$

Le théorème suivant nous donne que l'énergie magnétique est petite comparée avec le terme dominant dans (1.3.8).

Théorème 1.3.4 (Estimation de l'énergie magnétique.). *Avec les notations et l'hypothèse du Théorème 1.3.1, l'énergie magnétique du minimiseur satisfait*

$$(\kappa H)^2 \int_\Omega |\operatorname{curl} \mathbf{A} - B_0|^2 \, dx = o\Big(\kappa H \ln \frac{\kappa}{H}\Big) \,, \qquad \textit{lorsque } \kappa \longrightarrow +\infty \,. \tag{1.3.11}$$

Ce théorème a été prouvé par Helffer et Kachmar [27] quand H vérifie (1.3.9).

Nous allons donner un développement asymptotique de l'énergie locale du minimiseur $\mathcal{E}(\psi, \mathbf{A}; \mathcal{D})$ définie dans (1.2.8) où $\mathcal{D} \subset \Omega$ est un ensemble ouvert.

Théorème 1.3.5. *Supposons que B_0 vérifie* (1.1.3) *et $\mathcal{D}$ un ouvert régulier tel que $\overline{\mathcal{D}} \subset \Omega$, nous avons,*

1. *Si $(\psi, \mathbf{A})$ est un minimiseur de* (1.1.2) *et $H(\kappa)$ vérifie* (1.3.1)*, alors,*

$$\mathcal{E}(\psi, \mathbf{A}, \mathcal{D}) \geq \kappa^2 \int_{\mathcal{D}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx + o\Big(\kappa H \ln \frac{\kappa}{H}\Big) \,, \quad \textit{lorsque } \kappa \longrightarrow +\infty \,. \tag{1.3.12}$$

2. *Si $(\psi, \mathbf{A})$ est un minimiseur de* (1.1.2) *et $H(\kappa)$ satisfait*

$$C^1_{min} \kappa^{\frac{3}{5}} \leq H \ll \kappa \qquad \text{lorsque } \kappa \longrightarrow +\infty \,, \tag{1.3.13}$$

où C^1_{min} est une constante positive, alors,

$$\mathcal{E}(\psi, \mathbf{A}, \mathcal{D}) \leq \kappa^2 \int_{\mathcal{D}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx + o\Big(\kappa H \ln \frac{\kappa}{H}\Big) \,, \quad \textit{lorsque } \kappa \longrightarrow +\infty \,. \tag{1.3.14}$$

Helffer et Kachmar ont étudié le même problème quand H vérifie (1.3.9). Pour être plus précis, le reste sous la condition (1.3.9) était $\dfrac{\kappa^3}{H} o(1)$, il devient maintenant $\kappa H \ln \frac{\kappa}{H} o(1)$.

Dans le théorème suivant, pour des champs appliqués qui vérifient (1.3.1) et pour des configurations (ψ, A) d'énergie minimale, on définit des vortex en s'inspirant des méthodes de Sandier et Serfaty [46] qui nous aident à obtenir des informations sur la distribution de la vorticité dans Ω. Partant de là, on peut associer à (ψ, A) une mesure de vorticité

$$\mu_\kappa = \frac{2\pi}{\kappa H} \sum_{i=1}^{m} d_i \delta_{a_i} \,,$$

où les paires $(a_i, d_i)_i$ sont les positions et les degrés des vortex de (ψ, A).

Théorème 1.3.6. *Avec les notations et l'hypothèse du Théorème 1.3.1. Il existe $m = m(\kappa)$ disques disjoints $(D_i(a_i, r_i))_{i=1}^m$ dans Ω tels que, lorsque $\kappa \longrightarrow +\infty$,*

1. $\sum_{i=1}^m r_i \leq (\kappa H)^{\frac{1}{2}} \left(\ln \frac{\kappa}{H}\right)^{-\frac{7}{4}} \int_\Omega \frac{1}{\sqrt{|B_0(x)|}} \, dx \, (1 + o(1))$.

2. $|\psi| \geq \frac{1}{2}$ *sur* $\cup_i \partial D_i$.

3. *Si* $d_i = \deg\left(\frac{\psi}{|\psi|}, \partial D_i\right)$ *désignet les degrés des vortex de* $\frac{\psi}{|\psi|}$ *sur* ∂D_i*, alors lorsque* $\kappa \longrightarrow +\infty$

$$\mu_\kappa = \frac{2\pi}{\kappa H}\sum_{i=1}^{m} d_i \delta_{a_i} \longrightarrow B_0(x)\,dx \qquad \text{et} \qquad |\mu_\kappa| = \frac{2\pi}{\kappa H}\sum_{i=1}^{m} |d_i|\delta_{a_i} \longrightarrow |B_0(x)|\,dx\,,$$

au sens de la convergence faible[3]*, où* dx *est la mesure de Lebesgue de* $\mathbb{R}^2$ *restreint à* Ω *.*

Le théorème 1.3.6 nous donne que si $(\psi, \mathbf{A})$ est un minimiseur et B_0 un champ magnétique variable vérifie (1.1.3), alors ψ a des vortex qui sont distribués partout dans Ω mais avec une densité non uniforme.
Mentionnons ici que dans [46], Sandier et Serfaty ont obtenu (lorsque H vérifie (1.3.4) et B_0 est constant) que ψ a des vortex qui sont distribués uniformement dans Ω.

1.3.2 Méthodes de démonstration.

Une première étape du travail est d'analyser l'état fondamental avec conditions aux limites de Neumann et avec conditions aux limites de Dirichlet. Nous démontrons l'existence d'une constante positive C, telle que si $R > 1$ et $0 < b < 1$ alors :

$$e_D(b, R) \leq e_N(b, R) + C\,R\,b^{\frac{1}{2}}\,.$$

De plus, à l'aide de la fonction $\hat{f}$ introduite en (1.2.2) nous avons décrit le comportement asymptotique du $e_D(b, R)$, ce qui nous permet (après avoir subdiviser le domaine Ω en carrés de taille ℓ) en utilisant l'inégalité ci dessus de minorer l'énergie dans chaque carré. Nous utilisons dans chaque carré des estimations qui nous permettent de montrer que le champ B_0 peut être considéré comme constant.

Dans la **majoration de l'énergie**, la méthode consiste à construire dans chaque carré des tests de configurations avec une condition aux limites de Dirichlet en utilisant le résultat de Proposition 3.2.4 qui contient la majoration de l'état fondamental $e_{\mathcal{D}}$.
Notons que dans les deux estimations (majoration et minoration), nous approximons l'énergie en excluant les carrés qui rencontrent $|B_0| < \epsilon$.

Dans une autre partie nous montrons que l'énergie magnétique est dans l'erreur en utilisant les résultats sur l'énergie de l'état fondamental et nous déterminons une version locale de l'énergie. Ceci nous donne l'asymptotique de l'énergie de l'état fondamental dans un sous-domaine.

Dans la dernière partie, nous nous inspirons des méthodes de Sandier et Serfaty sur la distribution des vortex dans Ω. À nouvau nous resubdivisons chaque carré en M^2 carrés parmi lesquels on distingue entre bons et mauvais carrés, ceci nécessitant une optimisation fine des paramètres. Pour les détails nous renvoyons le lecteur à la Section 3.7. Nous démontrons que le nombre des mauvais carrés sont relativement petits, comparé à celui des bons carrés. En utilisant

[3] μ_κ est faiblement convergente vers μ c'est-à-dire :

$$\mu_\kappa(f) \longrightarrow \mu(f)\,, \qquad \forall f \in C_0(\Omega)\,.$$

le résultat de la Proposition 3.7.3 qui a prouvé par Sandier et Serfaty, nous obtenons que les minimiseurs ont des vortex qui sont distribués dans Ω avec une densité non uniforme.

1.4 Énergie pour un modèle de Ginzburg-Landau avec un champ magnétique variable et un terme de "pinning"

Nous allons étudier un modèle de Ginzburg-Landau avec un champ magnétique variable et avec *pinning*. Le *pinning* traduit l'existence d'impuretés dans le matériau, il peut avoir des conséquences importantes sur la localisation des vortex (sujet que nous n'abordons pas ici) et aussi sur la température critique et l'apparition de la supraconductivité. Il est modélisé (selon la littérature physique) par un terme de poids $a(x)$ qui correspond à l'inhomogénéité du matériau. L'énergie à étudier devient alors :

$$\mathcal{E}_{\kappa,H,a,B_0}(\psi,\mathbf{A}) = \int_\Omega \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 + \frac{\kappa^2}{2}(a(x,\kappa) - |\psi|^2)^2 \right) dx + \kappa^2 H^2 \int_\Omega |\operatorname{curl}\mathbf{A} - B_0|^2\, dx\,, \tag{1.4.1}$$

où $a(\kappa,x)$ peut dépendre de l'échelle κ. La fonction $a(x,\kappa)$ est réelle, définie sur $\overline{\Omega}\times[\kappa_0,+\infty)$ pour un certain $\kappa_0 > 0$ et satisfait les hypothèses suivantes :

(H_1)

$$\forall \kappa \geq \kappa_0\,, \quad a(\cdot,\kappa) \in C^1(\overline{\Omega})\,. \tag{1.4.2}$$

(H_2)

$$\sup_{x\in\overline{\Omega},\,\kappa\geq\kappa_0} |a(x,\kappa)| < +\infty\,. \tag{1.4.3}$$

(H_3)

$$\forall \kappa \geq \kappa_0\,, \quad \sup_{x\in\overline{\Omega}} |\nabla_x\, a(x,\kappa)| < +\infty\,. \tag{1.4.4}$$

(H_4) Il existe une constante positive C_1, telle que,

$$\forall \kappa \geq \kappa_0\,, \qquad \mathcal{L}\left(\partial\{a(x,\kappa) > 0\}\right) \leq C_1\,\kappa^{\frac{1}{2}}\,, \tag{1.4.5}$$

où $\mathcal{L}$ est la longueur de $\partial\{a(x,\kappa) > 0\}$ dans Ω. Pour les détails nous renvoyons le lecteur à la section 4.3 (plus précisement à l'équation 4.3.1).

L'hypothèse H_3 nous donne un contrôle uniforme sur l'oscillation de $a(\cdot,\kappa)$ qui sera précisé ultérieurement par une hypothèse sur

$$L(\kappa) = \sup_x |\nabla_x a(x,\kappa)|\,.$$

Si $(\psi,\mathbf{A}) \in H^1(\Omega,\mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ est un point critique de $\mathcal{E}_{\kappa,H,a,B_0}$, donc, $(\psi,\mathbf{A})$ est une solution faible du système constistué des équations suivantes, que nous appelons *'équations de Ginzburg-*

Landau',

$$\begin{cases} -(\nabla - i\kappa H\mathbf{A})^2\psi = \kappa^2(a(x,\kappa) - |\psi|^2)\psi & \text{dans} \quad \Omega \\ -\nabla^\perp \operatorname{rot}(\mathbf{A}-\mathbf{F}) = \dfrac{1}{\kappa H}\operatorname{Im}(\overline{\psi}\,(\nabla - i\kappa H\mathbf{A})\psi) & \text{dans} \quad \Omega \\ \nu\cdot(\nabla - i\kappa H\mathbf{A})\psi = 0 & \text{sur} \quad \partial\Omega \\ \operatorname{rot}\mathbf{A} = \operatorname{rot}\mathbf{F} & \text{sur} \quad \partial\Omega\,. \end{cases} \tag{1.4.6}$$

La fonctionelle $\mathcal{E}_{\kappa,H,a,B_0}$ a un point critique du type $(0,\mathbf{A})$ avec $\mathbf{A}\in H^1_{\mathrm{div}}(\Omega)$ tel que $\operatorname{rot}\mathbf{A} = B_0$. Un point critique ayant cette forme est appelé un état normal (ou une solution triviale). Il est donc naturel d'étudier si cet état normal est un minimiseur local de la fonctionnelle $\mathcal{E}_{\kappa,H,a,B_0}$ en présence d'un champ magnétique fort avec *pinning*. Le Hessien de $\mathcal{E}_{\kappa,H,a,B_0}$ en un état normal est donné par :

$$Hess_{(0,\mathbf{F})}[\phi,B] = \int_\Omega \left(|(\nabla - i\kappa H\mathbf{F})\phi|^2 - \kappa^2\, a(x,\kappa)|\phi|^2\right)\,dx + (\kappa H)^2\int_\Omega |\operatorname{rot}B|^2\,dx\,. \tag{1.4.7}$$

Nous sommes conduits alors à étudier la positivité de la forme quadratique

$$\mathcal{Q}^\Omega_{\kappa H\mathbf{F},-\kappa^2 a}(u) = \int_\Omega \left(|(\nabla - i\kappa H\mathbf{F})u|^2 - \kappa^2\, a(x,\kappa)|u|^2\right)\,dx \qquad (u\in H^1(\Omega))\,. \tag{1.4.8}$$

En observant que $\mathcal{Q}^\Omega_{\kappa H\mathbf{F},-\kappa^2 a}$ est semi-bornée inférieurement, i.e. qu'il existe une constante C telle que :

$$\int_\Omega \left(|(\nabla - i\kappa H\mathbf{F})u|^2 - \kappa^2\, a(x,\kappa)|u|^2\right)\,dx \geq -C\|u\|^2\,,$$

nous considérons la réalisation auto-adjointe définie par le théorème de Friedrichs. C'est l'opérateur de Schrödinger magnétique $P^\Omega_{\kappa H\mathbf{F},-\kappa^2 a}$ de domaine $D(P^\Omega_{\kappa H\mathbf{F},-\kappa^2 a})$:

$$\begin{aligned} P^\Omega_{\kappa H\mathbf{F},-\kappa^2 a} &= -(\nabla - i\kappa H\mathbf{F})^2\,, \\ D(P^\Omega_{\kappa H\mathbf{F},-\kappa^2 a}) &= \{u\in H^2(\Omega);\, \nu\cdot(\nabla - i\kappa H\mathbf{A})u_{|\partial\Omega} = 0\}\,. \end{aligned}$$

Notons $\mu_1(\kappa,H)$ le bas du spectre de l'opérateur $P^\Omega_{\kappa H\mathbf{F},-\kappa^2 a}$. À l'aide du principe du min-max, $\mu_1(\kappa,H)$ est défini par :

$$\mu_1(\kappa,H) = \inf_{\substack{\phi\in H^1(\Omega)\\ \phi\neq 0}} \left(\frac{\mathcal{Q}^\Omega_{\kappa H\mathbf{F},-\kappa^2 a}(\phi)}{\|\phi\|^2_{L^2(\Omega)}}\right)\,. \tag{1.4.9}$$

Beaucoup d'articles sont consacrés à l'estimation de minimum de l'énergie de Ginzburg-Landau avec *pinning*, la plupart de ces papiers ont étudiée l'influence du *pinning* sur la localisation des *vorticités*. Dans le cas où $B_0 = 0$ dans (1.4.6), l'influence du *pinning* a été étudié par Lassoued et Mironescu dans [37] et récemment par Michaël dans [39]. Le *pinning* (i.e. la fonction a) dans [37], est une fonction indépendante de κ, et est considéré dans [39] comme une fonction périodique dépendant de κ. La version magnétique de la fonctionnelle dans [37] a été étudiée dans [31, 33].

Dans [4], Aftalion, Sandier et Serfaty ont considéré une fonction a régulière dépendant de κ qui satisfait :

- $L(\kappa) \ll \kappa H$.
- il existe une fonction $a(x)$ continue, une constante positive a_0 et, pout tout $\kappa \geq 0$, il existe deux fonctions $\sigma(\kappa) = o\left(\left(\ln \left|\ln \frac{1}{\kappa}\right|\right)^{-\frac{1}{2}}\right)$ et $\beta(x,\kappa) \geq 0$ telles que,

$$\min_{B(x,\sigma(\kappa))} \beta(x,\kappa) = 0\,, \qquad a(x,\kappa) = a(x) + \beta(x,\kappa)\,, \qquad \text{et} \qquad 0 < a_0 \leq a(x) \leq 1\,.$$

Cette étude contient le cas quand $a(x,\kappa) = a(x)$ ($\beta = 0$), mais elle contient aussi des cas avec des fonctions $\beta(\cdot,\kappa)$ dont l'oscillation en x peut croître avec κ.

La fonctionnelle $\mathcal{E}_{\kappa,H,a,B_0}$ dans (1.4.6) est proche des modèles de Bose-Einstein (voir [1, 2]).

1.4.1 Résultats principaux

Nous allons analyser comment le *pinning* apparaît dans l'asymptotique de l'énergie en présence d'un champ magnétique externe, variable et fort.

Nous nous concentrons sur le régime κ grand (i.e. $\kappa \to +\infty$) et nous étudions l'énergie de l'état fondamental de Ginzburg-Landau comme suit :

$$\widetilde{\mathrm{E}_{\mathrm{g}}}(\kappa, H, a, B_0) = \inf\left\{\mathcal{E}_{\kappa,H,a,B_0}(\psi,\mathbf{A}) \ : \ (\psi,\mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)\right\}. \tag{1.4.10}$$

Pour être plus précis nous donnons une estimation asymptotique de $\widetilde{\mathrm{E}_{\mathrm{g}}}$, qui est valable quand H vérifie (1.2.1). Le comportement de l'énergie $\widetilde{\mathrm{E}_{\mathrm{g}}}$ fait intervenir la fonction $\hat{f}$ déjà définie dans (1.2.2).

Théorème 1.4.1. *Supposons que les hypothèses* $(H_1) - (H_4)$ *sont satisfaites et que* H *vérifie* (1.2.1)*, et*

$$L(\kappa) = \mathcal{O}(\kappa^{\frac{1}{2}}) \qquad \text{lorsque } \kappa \to +\infty\,. \tag{1.4.11}$$

L'énergie de l'état fondamental dans (1.4.10) *satisfait, lorsque* $\kappa \to +\infty$,

$$\widetilde{\mathrm{E}_{\mathrm{g}}}(\kappa, H, a, B_0) = \kappa^2 \int_{\{a(x,\kappa)>0\}} a(x,\kappa)^2\, \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) dx + \frac{\kappa^2}{2}\int_{\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\, dx + o\left(\kappa^2\right)\,. \tag{1.4.12}$$

Notons que quand $\Omega \cap \{a(x,\kappa) > 0\} = \varnothing$, nous obtenons directement du (1.4.1)

$$\mathcal{E}_{\kappa,H,a,B_0}(\psi,\mathbf{A}) \geq \frac{\kappa^2}{2}\int_\Omega a(x,\kappa)^2\, dx = \mathcal{E}_{\kappa,H,a,B_0}(0,\mathbf{F})\,.$$

Par conséquent le minimiseur de $\mathcal{E}_{\kappa,H,a,B_0}$ est l'état normal. En termes physiques, ce cas correspond au cas où nous sommes au dessus de la température critique.

Les hypothèses du Théorème 1.4.1 couvrent le cas où la fonction a est constante et égale à 1, qui a été étudié dans [5] (voir (1.2.4)) lorsque H vérifie (1.2.1).

Le théorème suivant nous donne une estimation de l'énergie magnétique.

Théorème 1.4.2 (Estimation de l'énergie magnétique)**.** *Avec les notations et les hypothèses du Théorème 1.4.1, nous avons*

$$(\kappa H)^2 \int_\Omega |\operatorname{curl} \mathbf{A} - B_0|^2\, dx = o(\kappa^2)\,, \qquad \text{lorsque } \kappa \longrightarrow +\infty\,. \tag{1.4.13}$$

Nous introduisons l'énergie locale de $(\psi, \mathbf{A})$ dans un domaine $\overline{\mathcal{D}} \subset \Omega$:

$$\mathcal{E}_0(\psi, \mathbf{A}; a, \mathcal{D}) = \int_{\mathcal{D}} |(\nabla - i\kappa H \mathbf{A})\psi|^2\, dx + \frac{\kappa^2}{2} \int_{\mathcal{D}} (a(x,\kappa) - |\psi|^2)^2\, dx\,. \tag{1.4.14}$$

Le théorème suivant, nous donne une estimation de l'énergie locale $\mathcal{E}_0(\psi, \mathbf{A}; a, \mathcal{D})$ et aussi un comportement asymptotique de la norme L^4 dans $\mathcal{D}$ du paramètre d'ordre ψ, quand $(\psi, \mathbf{A})$ est un minimiseur global.

Théorème 1.4.3. *Avec les notations et les hypothèses du Théorème 1.4.1, si* $(\psi, \mathbf{A})$ *est un minimiseur de* (1.4.1) *et* $\mathcal{D}$ *est un ensemble régulier tel que* $\overline{\mathcal{D}} \subset \Omega$*, alors, lorsque* $\kappa \to +\infty$*,*

- **Estimation de l'énergie locale** *:*

$$\begin{aligned}\mathcal{E}_0(\psi, \mathbf{A}; a, \mathcal{D}) = \kappa^2 \int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2\, \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) dx \\ + \frac{\kappa^2}{2} \int_{\mathcal{D}\cap\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\, dx + o\left(\kappa^2\right)\,.\end{aligned} \tag{1.4.15}$$

- **Concentration du paramètre d'ordre** *:*

$$\int_{\mathcal{D}} |\psi(x)|^4\, dx = -\int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2 \left\{2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) - 1\right\} dx + o\left(1\right)\,. \tag{1.4.16}$$

La formule (1.4.16) indique que ψ est localisé asymptotiquement dans la région où $a > 0$. Quand $a(x,\kappa) = 1$, le Théorème 1.4.3 a été prouvé dans [5].

Nous allons maintenant étudier le troisième champ crtitique, i.e. le champ au-dessus duquel l'état normal $(0, \mathbf{F})$ est le seul point critique de la fonctionnelle dans (1.4.6). Nous définissons les trois ensembles suivants :

$$\mathcal{N}^{\mathrm{cp}}(\kappa) = \{H > 0 : \mathcal{E}_{\kappa,H,a,B_0} \text{ a un point critique non normal}\}\,, \tag{1.4.17}$$

$$\mathcal{N}(\kappa) = \{H > 0 : \mathcal{E}_{\kappa,H,a,B_0} \text{ a un minimiseur non normal}\}\,, \tag{1.4.18}$$

et

$$\mathcal{N}^{\mathrm{loc}}(\kappa) = \{H > 0 : \mu_1(\kappa, H) < 0\}\,. \tag{1.4.19}$$

Ici, $\mu_1(\kappa, H)$ a été défini dans (1.4.9).
Nous renvoyons à [11, 32, 38, 47] pour des contributions précédentes.

Nous introduisons les champs critiques suivants (cf. e.g.[16, 35]).

$$\overline{H}^{cp}_{C_3}(\kappa) = \sup \mathcal{N}^{cp}(\kappa)\,, \qquad \underline{H}^{cp}_{C_3}(\kappa) = \inf\left(\mathbb{R}_+ \setminus \mathcal{N}^{cp}(\kappa)\right), \tag{1.4.20}$$

$$\overline{H}_{C_3}(\kappa) = \sup \mathcal{N}(\kappa)\,, \qquad \underline{H}_{C_3}(\kappa) = \inf\left(\mathbb{R}_+ \setminus \mathcal{N}(\kappa)\right), \tag{1.4.21}$$

$$\overline{H}^{loc}_{C_3}(\kappa) = \sup \mathcal{N}^{loc}(\kappa)\,, \qquad \underline{H}^{loc}_{C_3}(\kappa) = \inf\left(\mathbb{R}_+ \setminus \mathcal{N}^{loc}(\kappa)\right). \tag{1.4.22}$$

Au dessous du $\underline{H}_{C_3}$, les états normaux perdront leur stabilité, et au dessus du $\overline{H}_{C_3}$, l'état normal est (après une transformation de jauge) le seul point critique de la fonctionnelle dans (1.4.6).

Notre objectif est de determiner le comportement asymptotique de tous les champs critiques lorsque $\kappa \longrightarrow +\infty$.

Il s'agit détudier des quantité spectrales liées à trois modèles qui dépendent de Γ (Γ étant vide ou non). Nous introduisons

$$\Theta_0 = \inf_{\xi\in\mathbb{R}} \mu(\xi)\,,$$

où μ est la valeur propre de l'opérateur

$$\mathfrak{h}^{N,\xi} := -\frac{d^2}{dt^2} + (t+\xi)^2 \qquad \text{dans } L^2(\mathbb{R}_+)\,,$$

avec la condition de Neumann au bord $u'(0) = 0$.

Théorème 1.4.4. *Supposons que $B_0 > 0$ dans $\overline{\Omega}$ et que $a \in C^1(\overline{\Omega})$ vérifie $\{a > 0\} \neq \varnothing$. Alors, lorsque $\kappa \longrightarrow +\infty$, tous les champs critiques satisfont :*

$$H_{C_3}(\kappa) = \max\left(\sup_{x\in\Omega} \frac{a(x)}{|B_0(x)|}, \sup_{x\in\partial\Omega} \frac{a(x)}{\Theta_0|B_0(x)|}\right)\kappa + \mathcal{O}(\kappa^{\frac{1}{2}})\,. \tag{1.4.23}$$

Nous introduisons,

$$\lambda_0 = \inf_{\tau\in\mathbb{R}} \lambda(\tau)\,, \tag{1.4.24}$$

où $\lambda(\tau)$ est la valeur propre de l'opérateur autoadjoint :

$$M(\tau) = -\frac{d^2}{dt^2} + \frac{1}{4}(t^2 + 2\tau)^2 \qquad \text{in } L^2(\mathbb{R})\,. \tag{1.4.25}$$

Nous considérons, pour tous $\theta \in (0,\pi)$ le bas du spectre $\lambda(\mathbb{R}^2_+, \theta)$ de l'opérateur

$$P^{\mathbb{R}^2_+}_{\mathbf{A}_{\mathrm{app},\theta},0} \quad \text{with} \quad \mathbf{A}_{\mathrm{app},\theta} = -\left(\frac{x_2^2}{2}\cos\theta, \frac{x_1^2}{2}\sin\theta\right). \tag{1.4.26}$$

Théorème 1.4.5. *Supposons que $\Gamma = \{x : B_0(x) = 0\} \neq \varnothing$, que* (1.1.3) *est satisfaite et que $a \in C^1(\overline{\Omega})$ vérifie $\{a > 0\} \neq \varnothing$. Lorsque $\kappa \longrightarrow +\infty$, les six champs critiques dans* (1.4.20)-(1.4.22) *satisfont :*

$$H_{C_3}(\kappa) = \max\left(\sup_{x\in\Gamma\cap\overline{\Omega}} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|}, \sup_{x\in\Gamma\cap\partial\Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+, \theta(x))^{\frac{3}{2}}|\nabla B_0(x)|}\right)\kappa^2 + \mathcal{O}\left(\kappa^{\frac{11}{6}}\right).$$

Ici, $\theta(x)$ désigne l'angle entre $\nabla B_0(x)$ et le vecteur normal $-\nu(x)$.

1.4.2 Méthodes de démonstration

Notre but principal est de déterminer des estimations de l'énergie de l'état fondamental en présence d'un poids qui peut changer de signe et de donner une étude détaillée du troisième champ critique H_{C_3} quand a ne dépend pas du paramètre κ, qui peut être défini de plusieurs façons (voir (1.4.20)-(1.4.22)), mais qui correspond à la transition entre l'état où il y a supraconductivité de surface au bord du domaine, et l'état normal.

Dans l'estimation de l'énergie, les techniques utilisées dans la démonstration sont inspirées de celles de [5] et [6] (où le cas $a(x,\kappa)=1$ a été traité). Au niveau technique, l'approche utilisée dans les preuves est différente de celles de [5, 18, 44] puisque nous n'utilisons pas les *estimations elliptiques uniformes*. Ces estimations sont utilisées fréquemment dans les articles sur la fonctionnelle de Ginzburg-Landau (voir [14]) avec un pinning constant. Elles sont d'abord apparues dans [35] et ont été étendues au régime complet dans [15]. Plus précisement nous donnons une généralisation de la formule (1.2.4), où l'intégrand doit être remplacer par $a^2(x,\kappa)\,\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)$ dans la région où $a(x,\kappa)>0$, et par $\frac{a(x,\kappa)^2}{2}$ ailleurs. De plus nous montrons que les vortex sont localisés dans la région où le pinning a est positif.

L'étude asymptotique du troisième champ critique H_{C_3}, nous conduit lorsque le champ magnétique appliqué B_0 est constant à l'analyse spectrale de l'opérateur de Gennes :

$$\mathfrak{h}^{N,\xi} := -\frac{d^2}{dt^2} + (t+\xi)^2 \qquad \text{dans } L^2(\mathbb{R}_+)\,,$$

avec conditions aux limites de Neumann $u'(0)=0$.
Dans le cas où le champ magnétique variable s'annule sur une courbe régulière, nous sommes conduit à faire l'analyse spectrale de l'opérateur de Montgomery :

$$M(\tau) = -\frac{d^2}{dt^2} + \frac{1}{4}(t^2+2\tau)^2 \qquad \text{dans } L^2(\mathbb{R})\,,$$

et de l'opérateur de Pan et Kwek :

$$-(\nabla - i\mathbf{A}_{\mathrm{app},\theta})^2 \qquad \text{dans } L^2(\mathbb{R}_+)\,,$$

où

$$\mathbf{A}_{\mathrm{app},\theta} = -\left(\frac{x_2^2}{2}\cos\theta, \frac{x_1^2}{2}\sin\theta\right) \qquad \text{et} \qquad 0\le\theta<\pi\,.$$

Nous proposons six définitions sur H_{C_3}, selon qu'on se place du point de vue de l'unicité de l'état normal comme point critiquede l'énergie $\mathcal{E}_{\kappa,H,a,B_0}$, du caractère minimisant de l'état normal, ou de sa stabilité linéaire. (plus précisement voir (1.4.20), (1.4.21) et (1.4.22)).
Nous montrons dans les Théorèmes 1.4.4 et 1.4.5 que ces six définitions ont la même valeur de H_{C_3} lorsque $\kappa\to+\infty$.

Chapitre 2

The ground state energy of the two dimensional Ginzburg-Landau functional with variable magnetic field

We consider the Ginzburg-Landau functional with a variable applied magnetic field in a bounded and smooth two dimensional domain. We determine an accurate asymptotic formula for the minimizing energy when the Ginzburg-Landau parameter and the magnetic field are large and of the same order. As a consequence, it is shown how bulk superconductivity decreases in average as the applied magnetic field increases.

2.1 Introduction

2.1.1 The functional and main results

We consider a bounded open simply connected set $\Omega \subset \mathbb{R}^2$ with smooth boundary. We suppose that Ω models a superconducting sample submitted to an applied external magnetic field. The energy of the sample is given by the Ginzburg-Landau functional,

$$\mathcal{E}_{\kappa,H}(\psi,\mathbf{A}) \;=\; \int_\Omega \left[|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\right] dx \,+\, \kappa^2 H^2 \int_\Omega |\operatorname{curl}\mathbf{A} \,-\, B_0|^2\, dx\,. \tag{2.1.1}$$

Here κ and H are two positive parameters ; κ (the Ginzburg-Landau constant) is a material parameter and H measures the intensity of the applied magnetic field. The wave function (order parameter) $\psi \in H^1(\Omega;\mathbb{C})$ describes the superconducting properties of the material. The induced magnetic field is $\operatorname{curl}\mathbf{A}$, where the potential $\mathbf{A} \in H^1_{\rm div}(\Omega)$, with $H^1_{\rm div}(\Omega)$ is the space defined in (2.1.4) below. Finally, $B_0 \in C^\infty(\overline{\Omega})$ is the intensity of the external variable magnetic field and satisfies :

$$|B_0| + |\nabla B_0| > 0 \text{ in } \overline{\Omega}\,. \tag{2.1.2}$$

The assumption in (2.1.2) implies that for any open set ω relatively compact in Ω the set $\{x \in \omega, B_0(x) = 0\}$ will be either empty, or consists of a union of smooth curves. Let $\mathbf{F} : \Omega \to \mathbb{R}^2$ be the unique vector field such that,

$$\operatorname{div} \mathbf{F} = 0 \ \text{ and } \ \operatorname{curl} \mathbf{F} = B_0 \text{ in } \Omega\,, \quad \nu \cdot \mathbf{F} = 0 \text{ on } \partial\Omega. \tag{2.1.3}$$

The vector ν is the unit interior normal vector of $\partial\Omega$. The construction of $\mathbf{F}$ is recalled in the appendix. We define the space,

$$H^1_{\rm div}(\Omega) = \{\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2) \in H^1(\Omega)^2 \ : \ \operatorname{div} \mathbf{A} = 0 \text{ in } \Omega\,,\ \mathbf{A} \cdot \nu = 0 \text{ on} \,\partial\Omega\,\}. \tag{2.1.4}$$

Critical points $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\rm div}(\Omega)$ of $\mathcal{E}_{\kappa,H}$ are weak solutions of the Ginzburg-Landau equations,

$$\begin{cases} -(\nabla - i\kappa H\mathbf{A})^2\psi = \kappa^2(1 - |\psi|^2)\psi & \text{in} \quad \Omega \\ -\nabla^\perp \operatorname{curl}(\mathbf{A} - \mathbf{F}) = \dfrac{1}{\kappa H} \operatorname{Im}(\overline{\psi}\,(\nabla - i\kappa H\mathbf{A})\psi) & \text{in} \quad \Omega \\ \nu \cdot (\nabla - i\kappa H\mathbf{A})\psi = 0 & \text{on} \quad \partial\Omega \\ \operatorname{curl} \mathbf{A} = \operatorname{curl} \mathbf{F} & \text{on} \quad \partial\Omega\,. \end{cases} \tag{2.1.5}$$

Here, $\operatorname{curl} \mathbf{A} = \partial_{x_1}\mathbf{A}_2 - \partial_{x_2}\mathbf{A}_1$ and $\nabla^\perp \operatorname{curl} \mathbf{A} = (\partial_{x_2}(\operatorname{curl} \mathbf{A}), -\partial_{x_1}(\operatorname{curl} \mathbf{A}))$. If $\operatorname{div} \mathbf{A} = 0$, then $\nabla^\perp \operatorname{curl} \mathbf{A} = \Delta\mathbf{A}$. In this paper, we study the ground state energy defined as follows :

$$\mathrm{E_g}(\kappa, H) = \inf \left\{\mathcal{E}_{\kappa,H}(\psi, \mathbf{A}) \ : \ (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\rm div}(\Omega)\right\}. \tag{2.1.6}$$

More precisely, we give an asymptotic estimate which is valid in the simultaneous limit $\kappa \to \infty$ and $H \to \infty$ in such a way that $\frac{H}{\kappa}$ remains asymptotically constant. The behavior of $\mathrm{E_g}(\kappa, H)$ involves an auxiliary function $g : [0, \infty) \to [-\frac{1}{2}, 0]$ introduced in [45] whose definition will be recalled in (2.2.5) below. The function g is increasing, continuous, $g(b) = 0$ for all $b \geq 1$ and $g(0) = -\frac{1}{2}$.

Theorem 2.1.1. *Let* $0 < \Lambda_{\min} < \Lambda_{\max}$. *Under Assumption* (2.1.2), *there exist positive constants* C, κ_0 *and* $\tau_0 \in (1, 2)$ *such that if*

$$\kappa_0 \leq \kappa, \qquad \Lambda_{\min} \leq \frac{H}{\kappa} \leq \Lambda_{\max}\,,$$

then the ground state energy in (2.1.6) *satisfies,*

$$\left|\mathrm{E_g}(\kappa, H) - \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}\,|B_0(x)|\right)\,dx\right| \leq C\kappa^{\tau_0}\,. \tag{2.1.7}$$

Theorem 2.1.1 was proved in [45] when the magnetic field is constant ($B_0(x) = 1$). However, the estimate of the remainder is not explicitly given in [45].

The approach used in the proof of Theorem 2.1.1 is slightly different from the one in [45], and is closer to that in [17] which studies the same problem when $\Omega \subset \mathbb{R}^3$ and B_0 constant.

Corollary 2.1.2. *Suppose that the assumptions of Theorem 2.1.1 are satisfied. Then the magnetic energy of the minimizer satisfies, for some positive constant C,*

$$(\kappa H)^2 \int_\Omega |\operatorname{curl} \mathbf{A} - B_0|^2\, dx \leq C\kappa^{\tau_0}\,. \tag{2.1.8}$$

Remark 2.1.3. The value of τ_0 depends on the properties of B_0 : we find $\tau_0 = \frac{7}{4}$ when B_0 does not vanish in $\overline{\Omega}$ and $\tau_0 = \frac{15}{8}$ in the general case.

Theorem 2.1.4. *Suppose the assumptions of Theorem 2.1.1 are satisfied. There exist positive constants C, κ_0 and a negative constant $\tau_1 \in (-1,0)$ such that, if $\kappa \geq \kappa_0$, and D is regular set such that $\overline{D} \subset \Omega$, then the following is true.*

1. *If $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\rm div}(\Omega)$ is a solution of (2.1.5), then,*

$$\frac{1}{2}\int_D |\psi|^4\, dx \leq -\int_D g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + C\kappa^{\tau_1}\,. \tag{2.1.9}$$

2. *If $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\rm div}(\Omega)$ is a minimizer of (2.1.1), then,*

$$\left|\int_D |\psi|^4\, dx + 2\int_D g\left(\frac{H}{\kappa}|B_0(x)|\right) dx\right| \leq C\kappa^{\tau_1}\,. \tag{2.1.10}$$

Remark 2.1.5. The value of τ_1 depends on the properties of B_0 : we find $\tau_1 = -\frac{1}{4}$ when B_0 does not vanish in $\overline{\Omega}$ and $\tau_1 = -\frac{1}{8}$ in the general case.

2.1.2 Discussion of main result :

If $\{x \in \overline{\Omega} : B_0(x) = 0\} \neq \varnothing$ and $H = b\kappa$, $b > 0$, then $g\left(\frac{H}{\kappa}|B_0(x)|\right) \neq 0$ in $D = \left\{x \in \Omega : \frac{H}{\kappa}|B_0(x)| < 1\right\}$, and $|D| \neq 0$. Consequently, for κ sufficiently large, the restriction of ψ on D is not zero in $L^2(\Omega)$. This is a significant difference between our result and the one for constant magnetic field. When the magnetic field is a non-zero constant, then (see [14]), there is a universal constant $\Theta_0 \in (\frac{1}{2}, 1)$ such that, if $H = b\kappa$ and $b > \Theta_0^{-1}$, then $\psi = 0$ in $\overline{\Omega}$. Moreover, in the same situation, when $H = b\kappa$ and $1 < b < \Theta_0^{-1}$, then ψ is small every where except in a thin tubular neighborhood of $\partial\Omega$ (see [48]). Our result goes in the same spirit as in [47], where the authors established under the Assumption (2.1.2) that when $H = b\kappa^2$ and $b > b_0$, then $\psi = 0$ in $\overline{\Omega}$. (b_0 is a constant).

2.1.3 Notation.

Throughout the paper, we use the following notation :

- We write $\mathcal{E}$ for the functional $\mathcal{E}_{\kappa,H}$ in (2.1.1).

- The letter C denotes a positive constant that is independent of the parameters κ and H, and whose value may change from a formula to another.
- If $a(\kappa)$ and $b(\kappa)$ are two positive functions, we write $a(\kappa) \ll b(\kappa)$ if $a(\kappa)/b(\kappa) \to 0$ as $\kappa \to \infty$.
- If $a(\kappa)$ and $b(\kappa)$ are two functions with $b(\kappa) \neq 0$, we write $a(\kappa) \sim b(\kappa)$ if $a(\kappa)/b(\kappa) \to 1$ as $\kappa \to \infty$.
- If $a(\kappa)$ and $b(\kappa)$ are two positive functions, we write $a(\kappa) \approx b(\kappa)$ if there exist positive constants c_1, c_2 and κ_0 such that $c_1 b(\kappa) \leq a(\kappa) \leq c_2 b(\kappa)$ for all $\kappa \geq \kappa_0$.
- If $x \in \mathbb{R}$, we let $[x]_+ = \max(x, 0)$.
- Given $R > 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$, we denote by $Q_R(x) = (-R/2 + x_1, R/2 + x_1) \times (-R/2 + x_2, R/2 + x_2)$ the square of side length R centered at x.
- We will use the standard Sobolev spaces $W^{s,p}$. For integer values of s these are given by

$$W^{n,p}(\Omega) := \left\{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq n \right\}.$$

- Finally we use the standard symbol $H^n(\Omega) = W^{n,2}(\Omega)$.

2.2 The limiting energy

2.2.1 Two-dimensional limiting energy

Given a constant $b \geq 0$ and an open set $\mathcal{D} \subset \mathbb{R}^2$, we define the following Ginzburg-Landau energy,

$$G^\sigma_{b,\mathcal{D}}(u) = \int_{\mathcal{D}} \left(b|(\nabla - i\sigma \mathbf{A}_0)u|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) dx\,, \qquad \forall\, u \in H^1_0(\mathcal{D}). \tag{2.2.1}$$

Here $\sigma \in \{-1, +1\}$ and $\mathbf{A}_0$ is the canonical magnetic potential,

$$\mathbf{A}_0(x) = \frac{1}{2}(-x_2, x_1)\,, \qquad \forall\, x = (x_1, x_2) \in \mathbb{R}^2, \tag{2.2.2}$$

that satisfies :

$$\operatorname{curl} \mathbf{A}_0 = 1 \;\text{ in } \mathbb{R}^2\,.$$

We write $Q_R = Q_R(0)$ and let

$$m_0(b, R) = \inf_{u \in H^1_0(Q_R;\mathbb{C})} G^{+1}_{b,Q_R}(u)\,. \tag{2.2.3}$$

Remark 2.2.1. As $G^{+1}_{b,\mathcal{D}}(u) = G^{-1}_{b,\mathcal{D}}(\overline{u})$, it is immediate that,

$$\inf_{u \in H^1_0(Q_R;\mathbb{C})} G^{-1}_{b,Q_R}(u) = \inf_{u \in H^1_0(Q_R;\mathbb{C})} G^{+1}_{b,Q_R}(u)\,. \tag{2.2.4}$$

The main part of the next theorem was obtained by Sandier-Serfaty [45] and Aftalion-Serfaty [3, Lemma 2.4]. However, the estimate in (2.2.7) is obtained by Fournais-Kachmar [18].

Theorem 2.2.2. *Let $m_0(b,R)$ be as defined in* (2.2.3).

1. *For all $b\geq 1$ and $R>0$, we have $m_0(b,R)=0$.*
2. *For any $b\in[0,\infty)$, there exists a constant $g(b)\leq 0$ such that,*
$$g(b)=\lim_{R\to\infty}\frac{m_0(b,R)}{|Q_R|}\qquad\text{and}\qquad g(0)=-\frac{1}{2}\,. \tag{2.2.5}$$
3. *The function $[0,+\infty)\ni b\mapsto g(b)$ is continuous, non-decreasing, concave and its range is the interval $[-\frac{1}{2},0]$.*
4. *There exists a constant $\alpha\in(0,\frac{1}{2})$ such that,*
$$\forall\ b\in[0,1]\,,\quad \alpha(b-1)^2\leq|g(b)|\leq\frac{1}{2}(b-1)^2\,. \tag{2.2.6}$$
5. *There exist constants C and R_0 such that,*
$$\forall\ R\geq R_0\,,\ \forall\ b\in[0,1]\,,\quad g(b)\leq\frac{m_0(b,R)}{R^2}\leq g(b)+\frac{C}{R}\,. \tag{2.2.7}$$

2.3 A priori estimates

The aim of this section is to give *a priori* estimates for solutions of the Ginzburg-Landau equations (2.1.5). These estimates play an essential role in controlling the errors resulting from various approximations. The starting point is the following L^∞-bound resulting from the maximum principle. Actually, if $(\psi,\mathbf{A})\in H^1(\Omega;\mathbb{C})\times H^1_{\rm div}(\Omega)$ is a solution of (2.1.5), then
$$\|\psi\|_{L^\infty(\Omega)}\leq 1\,. \tag{2.3.1}$$

The set of estimates below is proved in [15, Theorem 3.3 and Eq. 3.35] (see also [40] for an earlier version).

Theorem 2.3.1. *Let $\Omega\subset\mathbb{R}^2$ be bounded and smooth and $B_0\in C^\infty(\overline{\Omega})$.*

1. *For all $p\in(1,\infty)$ there exists $C_p>0$ such that, if $(\psi,\mathbf{A})\in H^1(\Omega,C)\times H^1_{\rm div}(\Omega)$ is a solution of* (2.1.5)*, then*
$$\|{\rm curl}(\mathbf{A}-\mathbf{F})\|_{W^{1,p}(\Omega)}\leq C_p\frac{1+\kappa H+\kappa^2}{\kappa H}\|\psi\|_{L^\infty(\Omega)}\|\psi\|_{L^2(\Omega)}\,. \tag{2.3.2}$$
2. *For all $\alpha\in(0,1)$ there exists $C_\alpha>0$ such that, if $(\psi,\mathbf{A})\in H^1(\Omega,C)\times H^1_{\rm div}(\Omega)$ is a solution of* (2.1.5)*, then*
$$\|{\rm curl}(\mathbf{A}-\mathbf{F})\|_{C^{0,\alpha}(\overline{\Omega})}\leq C_\alpha\frac{1+\kappa H+\kappa^2}{\kappa H}\|\psi\|_{L^\infty(\Omega)}\|\psi\|_{L^2(\Omega)}\,. \tag{2.3.3}$$

3. *For all* $p \in [2,\infty)$ *there exists* $C > 0$ *such that, if* $\kappa > 0$, $H > 0$ *and* $(\psi, \mathbf{A}) \in H^1(\Omega, C) \times H^1_{\rm div}(\Omega)$ *is a solution of* (2.1.5), *then*

$$\|(\nabla - i\kappa H\mathbf{A})^2\psi\|_p \leq \kappa^2 \|\psi\|_p\,, \tag{2.3.4}$$

$$\|(\nabla - i\kappa H\mathbf{A})\psi\|_2 \leq \kappa \|\psi\|_2\,, \tag{2.3.5}$$

$$\|\operatorname{curl}(\mathbf{A} - \mathbf{F})\|_{W^{1,p}(\Omega)} \leq \frac{C}{\kappa H}\|\psi\|_\infty \|(\nabla - i\kappa H\mathbf{A})\psi\|_p\,. \tag{2.3.6}$$

Remark 2.3.2. :

1. Using the $W^{k,p}$-regularity of the Curl-Div system [14, Appendix A, Proposition A.5.1], we obtain from (2.3.2),

$$\|\mathbf{A} - \mathbf{F}\|_{W^{2,p}(\Omega)} \leq C_p \frac{1 + \kappa H + \kappa^2}{\kappa H} \|\psi\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)}\,. \tag{2.3.7}$$

   The estimate is true for any $p \in [2, \infty)$.

2. Using the Sobolev embedding Theorem we get, for all $\alpha \in (0,1)$

$$\|\mathbf{A} - \mathbf{F}\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_\alpha \frac{1 + \kappa H + \kappa^2}{\kappa H} \|\psi\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)}\,. \tag{2.3.8}$$

3. Combining (2.3.5) and (2.3.6) (with $p = 2$) yields

$$\|\operatorname{curl}(\mathbf{A} - \mathbf{F})\|_{L^2(\Omega)} \leq \frac{C}{H} \|\psi\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)}\,. \tag{2.3.9}$$

Theorem 2.3.1 is needed in order to obtain the improved *a priori* estimates of the next theorem. Similar estimates are given in [40].

Theorem 2.3.3. *Suppose that* $0 < \Lambda_{\min} \leq \Lambda_{\max}$. *There exist constants* $\kappa_0 > 1$, $C_1 > 0$ *and for any* $\alpha \in (0,1)$, $C_\alpha > 0$ *such that, if*

$$\kappa \geq \kappa_0\,, \quad \Lambda_{\min} \leq \frac{H}{\kappa} \leq \Lambda_{\max}\,, \tag{2.3.10}$$

and $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\rm div}(\Omega)$ *is a solution of* (2.1.5), *then*

$$\|(\nabla - i\kappa H\mathbf{A})\psi\|_{C(\overline{\Omega})} \leq C_1\sqrt{\kappa H}\|\psi\|_{L^\infty(\Omega)}\,, \tag{2.3.11}$$

$$\|\mathbf{A} - \mathbf{F}\|_{H^2(\Omega)} \leq C_1 \left(\|\operatorname{curl}(\mathbf{A} - \mathbf{F})\|_{L^2(\Omega)} + \frac{1}{\sqrt{\kappa H}} \|\psi\|_{L^2(\Omega)} \|\psi\|_{L^\infty(\Omega)} \right), \tag{2.3.12}$$

$$\|\mathbf{A} - \mathbf{F}\|_{C^{0,\alpha}(\overline{\Omega})} \leq C_\alpha \left(\|\operatorname{curl}(\mathbf{A} - \mathbf{F})\|_{L^2(\Omega)} + \frac{1}{\sqrt{\kappa H}} \|\psi\|_{L^2(\Omega)} \|\psi\|_{L^\infty(\Omega)} \right). \tag{2.3.13}$$

Proof.
Proof of (2.3.11) **:** See [14, Proposition 12.4.4].
Proof of (2.3.12) **:**

Let $a = \mathbf{A} - \mathbf{F}$. Since $\operatorname{div} a = 0$ and $a \cdot \nu = 0$ on $\partial\Omega$, we get by regularity of the curl-div system (see Proposition 2.8.1),

$$\|a\|_{H^2(\Omega)} \leq C \| \operatorname{curl} a\|_{H^1(\Omega)} \,. \tag{2.3.14}$$

The second equation in (2.1.5) reads as follows,

$$-\nabla^\perp \operatorname{curl} a = \frac{1}{\kappa H} \operatorname{Im}(\overline{\psi}\, (\nabla - i\kappa H \mathbf{A})\psi)\,.$$

The estimates in (2.3.11) and (2.3.14) now give,

$$\|a\|_{H^2(\Omega)} \leq C \left(\| \operatorname{curl} a\|_{L^2(\Omega)} + \frac{1}{\sqrt{\kappa H}} \, \|\psi\|_{L^2(\Omega)} \|\psi\|_{L^\infty(\Omega)} \right)\,.$$

Proof of (2.3.13) **:**
This is a consequence of the Sobolev embedding of $H^2(\Omega)$ into $C^{0,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$ and (2.3.12). □

2.4 Energy estimates in small squares

If $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\rm div}(\Omega)$, we introduce the energy density,

$$e(\psi, \mathbf{A}) = |(\nabla - i\kappa H \mathbf{A})\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\,.$$

We also introduce the local energy of $(\psi, \mathbf{A})$ in a domain $D \subset \Omega$:

$$\mathcal{E}_0(u, \mathbf{A}; D) = \int_D e(\psi, \mathbf{A})\, dx\,. \tag{2.4.1}$$

Furthermore, we define the Ginzburg-Landau energy of $(\psi, \mathbf{A})$ in a domain $D \subset \Omega$ as follows,

$$\mathcal{E}(\psi, \mathbf{A}; D) = \mathcal{E}_0(\psi, \mathbf{A}; D) + (\kappa H)^2 \int_\Omega |\operatorname{curl}(\mathbf{A} - \mathbf{F})|^2\, dx\,. \tag{2.4.2}$$

If $D = \Omega$, we sometimes omit the dependence on the domain and write $\mathcal{E}_0(\psi, \mathbf{A})$ for $\mathcal{E}_0(\psi, \mathbf{A}; \Omega)$. We start with a lemma that will be useful in the proof of Proposition 2.4.2 below. Before we start to state the lemma, we define for all (ℓ, x_0) such that $\overline{Q_\ell(x_0)} \subset \Omega$,

$$\overline{B}_{Q_\ell(x_0)} = \sup_{x \in Q_\ell(x_0)} |B_0(x)|\,, \tag{2.4.3}$$

where B_0 is introduced in (2.1.2). Later x_0 will be chosen in a lattice of $\mathbb{R}^2$.

Lemma 2.4.1. *For any $\alpha \in (0,1)$. there exist positive constants C and κ_0 such that if* (2.3.10) *holds,* $0 < \delta < 1$, $\quad 0 < \ell < 1$*, and $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\rm div}(\Omega)$ is a critical point of* (2.1.1) *(i.e. a solution of* (2.1.5)*), then, for any square $Q_\ell(x_0)$ relatively compact in $\Omega \cap \{|B_0| > 0\}$,* *there exists* $\varphi \in H^1(\Omega)$*, such that,*

$$\mathcal{E}_0(\psi,\mathbf{A};Q_\ell(x_0)) \geq (1-\delta)\mathcal{E}_0(e^{-i\kappa H\varphi}\psi, \sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0), Q_\ell(x_0)) \\ - C\kappa^2\left(\delta^{-1}\ell^{2\alpha}+\delta^{-1}\ell^4\kappa^2+\delta\right)\int_{Q_\ell(x_0)}|\psi|^2\,dx\,, \quad (2.4.4)$$

where σ_ℓ *denotes the sign of* B_0 *in* $Q_\ell(x_0)$*.*

Proof.

Construction of φ **:**

Let $\phi_{x_0}(x) = \big(\mathbf{A}(x_0) - \mathbf{F}(x_0)\big)\cdot x$, where $\mathbf{F}$ is the magnetic potential introduced in (2.1.3). Using the estimate in (2.3.13), we get for all $x \in Q_\ell(x_0)$ and $\alpha \in (0,1)$,

$$\begin{aligned} |\mathbf{A}(x) - \nabla\phi_{x_0} - \mathbf{F}(x)| &= |(\mathbf{A}-\mathbf{F})(x) - (\mathbf{A}-\mathbf{F})(x_0)| \\ &\leq \|\mathbf{A}-\mathbf{F}\|_{C^{0,\alpha}}\cdot|x-x_0|^\alpha \\ &\leq C\frac{\sqrt{\lambda}}{\kappa H}\,\ell^\alpha\,, \end{aligned} \quad (2.4.5)$$

where

$$\lambda = (\kappa H)^2\left(\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|^2_{L^2(\Omega)} + \frac{1}{\kappa H}\|\psi\|^2_{L^2(\Omega)}\right).$$

Using the bound $\|\psi\|_\infty \leq 1$ and the estimate in (2.3.9), we get

$$\lambda \leq C\kappa^2\,, \quad (2.4.6)$$

which implies that

$$|\mathbf{A}(x) - \nabla\phi_{x_0}(x) - \mathbf{F}(x)| \leq C\,\frac{\ell^\alpha}{H}\,. \quad (2.4.7)$$

We estimate the energy $\mathcal{E}_0(\psi,\mathbf{A};Q_\ell(x_0))$ from below. We will need the function φ_0 introduced in Lemma 2.8.3 and satisfiying

$$|\mathbf{F}(x) - \sigma_\ell\overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0) - \nabla\varphi_0(x)| \leq C\ell^2 \qquad \text{in } Q_\ell(x_0).$$

Let

$$u = e^{-i\kappa H\varphi}\psi\,, \quad (2.4.8)$$

where $\varphi = \varphi_0 + \phi_{x_0}$.

Lower bound :

We start with estimating the kinetic energy from below as follows. For any $\delta \in (0,1)$, we write

$$\begin{aligned} &|(\nabla - i\kappa H\mathbf{A})\psi|^2 \\ &\quad= \Big|\Big(\nabla - i\kappa H(\sigma_\ell\overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0)+\nabla\varphi)\Big)\psi - i\kappa H\Big(\mathbf{A} - \sigma_\ell\overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0) - \nabla\varphi\Big)\psi\Big|^2 \\ &\quad\geq (1-\delta)\Big|\Big(\nabla - i\kappa H(\sigma_\ell\overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0)+\nabla\varphi)\Big)\psi\Big|^2 \\ &\qquad + (1-\delta^{-1})(\kappa H)^2\left|(\mathbf{A}-\nabla\phi_{x_0}-\mathbf{F})\psi + (\mathbf{F} - \sigma_\ell\overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0) - \nabla\varphi_0)\psi\right|^2. \end{aligned}$$

Using the estimates in (2.4.7), (2.8.3) and the assumption in (2.3.10), we get, for any $\alpha \in (0,1)$

$$
\begin{aligned}
|(\nabla - i\kappa H\mathbf{A})\psi|^2 \geq (1-\delta)\Big|\Big(\nabla - i\kappa H(\sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0) + \nabla\varphi)\Big)\psi\Big|^2 \\
- C\kappa^2\left(\delta^{-\frac{1}{2}}\ell^\alpha + \delta^{-\frac{1}{2}}\ell^2 H\right)^2 |\psi|^2 . \quad (2.4.9)
\end{aligned}
$$

Remembering the defintion of u in (2.4.8), then, we deduce the lower bound of $\mathcal{E}_0$,

$$
\begin{aligned}
&\mathcal{E}_0(\psi,\mathbf{A};Q_\ell(x_0)) \\
&\quad\geq \int_{Q_\ell(x_0)} \left[(1-\delta)|(\nabla - i\kappa H(\sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0))u|^2 - \kappa^2|u|^2 + \frac{\kappa^2}{2}|u|^4\right] dx \\
&\qquad\qquad - C\kappa^2\left(\delta^{-\frac{1}{2}}\ell^2\kappa + \delta^{-\frac{1}{2}}\ell^\alpha\right)^2 \int_{Q_\ell(x_0)} |\psi|^2 dx \\
&\quad\geq (1-\delta)\mathcal{E}_0(u, \sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0); Q_\ell(x_0)) \\
&\qquad\qquad - \widehat{C}\kappa^2\left(\delta^{-1}\ell^4\kappa^2 + \delta^{-1}\ell^{2\alpha} + \delta\right) \int_{Q_\ell(x_0)} |\psi|^2\, dx . \quad (2.4.10)
\end{aligned}
$$

This finishes the proof of Lemma 2.4.1. □

Proposition 2.4.2. *For all $\alpha \in (0,1)$, there exist positive constants C, ϵ_0 and κ_0 such that, if* (2.3.10) *holds, $\kappa \geq \kappa_0$, $\ell \in (0, \frac{1}{2})$, $\epsilon \in (0, \epsilon_0)$, $\ell^2\kappa^2\epsilon > 1$, $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ a critical point of* (2.1.1)*, and $\overline{Q_\ell(x_0)} \subset \Omega \cap \{|B_0| > \epsilon\}$, then*

$$
\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi,\mathbf{A};Q_\ell(x_0)) \geq g\left(\frac{H}{\kappa}\overline{B}_{Q_\ell(x_0)}\right)\kappa^2 - C\left(\ell^3\kappa^2 + \ell^{2\alpha-1} + (\ell\kappa\epsilon)^{-1} + \ell\epsilon^{-1}\right)\kappa^2 .
$$

Here $g(\cdot)$ is the function introduced in (2.2.5)*, and $\overline{B}_{Q_\ell(x_0)}$ is introduced in* (2.4.3)*.*

Proof.
Using the inequality $\|\psi\|_\infty \leq 1$ and (2.4.4) to obtain,

$$
\begin{aligned}
\mathcal{E}_0(\psi,\mathbf{A};Q_\ell(x_0)) \geq (1-\delta)\mathcal{E}_0(u, \sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0); Q_\ell(x_0)) \\
- C\kappa^2\left(\delta^{-1}\ell^4\kappa^2 + \delta^{-1}\ell^{2\alpha} + \delta\right)|Q_\ell(x_0)| , \quad (2.4.11)
\end{aligned}
$$

where u is defined in (2.4.8).
Let

$$
b = \frac{H}{\kappa}\overline{B}_{Q_\ell(x_0)} , \qquad R = \ell\sqrt{\kappa H \overline{B}_{Q_\ell(x_0)}} . \quad (2.4.12)
$$

Define the rescaled function,

$$
v(x) = u\left(\frac{\ell}{R}x + x_0\right) , \qquad \forall\, x \in Q_R . \quad (2.4.13)
$$

Remember that σ_ℓ denotes the sign of B_0 in $Q_\ell(x_0)$. The change of variable $y=\frac{R}{\ell}(x-x_0)$ gives :

$$
\begin{aligned}
&\mathcal{E}_0(u,\sigma_\ell\overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0)\,;Q_\ell(x_0))\\
&\qquad=\int_{Q_R}\left(\left|\left(\frac{R}{\ell}\nabla_y-i\sigma_\ell\frac{R}{\ell}\mathbf{A}_0(y)\right)v\right|^2-\kappa^2|v|^2+\frac{\kappa^2}{2}|v|^4\right)\frac{\ell}{R}dy\\
&\qquad=\int_{Q_R}\left(|(\nabla_y-i\sigma_\ell\mathbf{A}_0)v|^2-\frac{\kappa}{H\overline{B}_{Q_\ell(x_0)}}|v|^2+\frac{\kappa}{2H\overline{B}_{Q_\ell(x_0)}}|v|^4\right)dy\\
&\qquad=\frac{\kappa}{H\overline{B}_{Q_\ell(x_0)}}\int_{Q_R}b\left(|(\nabla_y-i\sigma_\ell\mathbf{A}_0)v|^2-|v|^2+\frac12|v|^4\right)dy\\
&\qquad=\frac1b\,G^{\sigma_\ell}_{b,Q_R}(v)\,.
\end{aligned}\tag{2.4.14}
$$

We still need to estimate from below the reduced energy $G^{\sigma_\ell}_{b,Q_R}(v)$. Since v is not in $H^1_0(Q_R)$, we introduce a cut-off function $\chi_R\in C_c^\infty(\mathbb{R}^2)$ such that

$$
0\le\chi_R\le1\quad\text{in }\mathbb{R}^2\,,\quad \operatorname{supp}\chi_R\subset Q_R\,,\quad \chi_R=1\quad\text{in }Q_{R-1}\,,\quad\text{and}\quad|\nabla\chi_R|\le M\text{ in }\mathbb{R}^2\,.\tag{2.4.15}
$$

The constant M is universal.
Let

$$
u_R=\chi_R\,v\,.\tag{2.4.16}
$$

We have,

$$
\begin{aligned}
G^{\sigma_\ell}_{b,Q_R}(v)&=\int_{Q_R}\left(b|(\nabla-i\sigma_\ell\mathbf{A}_0)v|^2-|v|^2+\frac12|v|^4\right)dx\\
&\ge\int_{Q_R}\left(b|\chi_R(\nabla-i\sigma_\ell\mathbf{A}_0)v|^2-|\chi_Rv|^2+\frac12|v|^4+(\chi_R^2-1)|v|^2\right)dx\\
&\ge G^{\sigma_\ell}_{b,Q_R}(\chi_Rv)-\int_{Q_R}(1-\chi_R^2)|v|^2dx-2\int_{Q_R}\Big|\langle(\nabla-i\sigma_\ell\mathbf{A}_0)\chi_Rv\,,\,\nabla\chi_Rv\rangle\Big|\,dy\,.
\end{aligned}\tag{2.4.17}
$$

Having in mind (2.4.13) and (2.4.8), we get,

$$
\Big|\Big(\nabla_y-i\sigma_\ell\mathbf{A}_0(y)\Big)v(y)\Big|=\frac{\ell}{R}\Big|\Big(\nabla_x-i\kappa H\sigma_\ell\overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0)\Big)u(x)\Big|\,.
$$

Using the estimate in (2.3.11), (2.4.7) and (2.8.3) we get,

$$
\begin{aligned}
\Big|\Big(\nabla_y-i\sigma_\ell\mathbf{A}_0(y)\Big)v(y)\Big|&\le\frac{\ell}{R}\Big|\Big(\nabla_x-i\kappa H\sigma_\ell\overline{B}_{Q_\ell(x_0)}(\mathbf{A}+\nabla\varphi)\Big)u(x)\Big|\\
&\qquad+\frac{\kappa H\ell}{R}\Big|(\mathbf{A}-\sigma_\ell\overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0)-\nabla\varphi)u(x)\Big|\\
&\le\frac{C_1\ell}{R}\left(\kappa+\kappa\ell^\alpha+\kappa^2\ell^2\right)\,.
\end{aligned}\tag{2.4.18}
$$

From the definition of u_R in (2.4.16) and χ_R in (2.4.15) we get,

$$|v| \leq 1\,. \tag{2.4.19}$$

Using (2.4.19), (2.4.18) and the definition of χ_R in (2.4.15), we get :

$$\begin{aligned}\int_{Q_R} \Big|\langle(\nabla - i\sigma_\ell \mathbf{A}_0)\chi_R v\,,\, \nabla\chi_R v\rangle\Big|\,dy &\leq \frac{C_1\ell}{R}\left(\kappa + \kappa\ell^\alpha + \kappa^2\ell^2\right)\int_{Q_R\setminus Q_{R-1}} |\nabla\chi_R|\,dx\\ &\leq C_1\left(\kappa\ell + \kappa\ell^{\alpha+1} + \kappa^2\ell^3\right)\,,\end{aligned} \tag{2.4.20}$$

and

$$\begin{aligned}\int_{Q_R}(1-\chi_R^2)|v|^2dx &\leq |Q_R\setminus Q_{R-1}|\\ &\leq R\,.\end{aligned} \tag{2.4.21}$$

Inserting (2.4.20) and (2.4.21) into (2.4.17), we get,

$$\begin{aligned}G^{\sigma_\ell}_{b,Q_R}(v) &\geq G^{\sigma_\ell}_{b,Q_R}(u_R) - C_2\left(\kappa\ell + \kappa\ell^{\alpha+1} + \kappa^2\ell^3\right)\\ &\geq G^{\sigma_\ell}_{b,Q_R}(u_R) - C_2\left(\kappa\ell + \kappa^2\ell^3\right).\end{aligned}$$

There are two cases :

Case 1 : $\sigma_\ell = +1$, when $B_0 > 0$, in $Q_\ell(x_0)$.

Case 2 : $\sigma_\ell = -1$, when $B_0 < 0$, in $Q_\ell(x_0)$.

In Case 1, after recalling the definition of $m_0(b,R)$ introduced in (2.2.3), where b is introduced in (2.4.12) we get,

$$G^{+1}_{b,Q_R}(v) \geq m_0(b,R) - C_2\left(\kappa\ell + \kappa^2\ell^3\right). \tag{2.4.22}$$

We get by collecting the estimates in (2.4.11)-(2.4.22) :

$$\begin{aligned}\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi,\mathbf{A};Q_\ell(x_0)) &\geq \frac{(1-\delta)}{b\ell^2}\left(m_0(b,R) - C_2\left(\kappa\ell + \kappa^2\ell^3\right)\right)\\ &\qquad\qquad - C\left(\delta^{-1}\ell^4\kappa^2 + \delta^{-1}\ell^{2\alpha} + \delta\right)\kappa^2\\ &\geq \frac{(1-\delta)}{b\ell^2}m_0(b,R) - r(\kappa)\,,\end{aligned} \tag{2.4.23}$$

where

$$r(\kappa) = C_3\left(\delta^{-1}\ell^4\kappa^4 + \delta^{-1}\ell^{2\alpha}\kappa^2 + \delta\kappa^2 + \frac{1}{b\ell^2}\left(\kappa\ell + \kappa^2\ell^3\right)\right). \tag{2.4.24}$$

Theorem 2.2.2 tells us that $m_0(b,R) \geq R^2 g(b)$ for all $b \in [0,1]$ and R sufficiently large. Here $g(b)$ is introduced in (2.2.5). Therefore, we get from (2.4.23) the estimate,

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi,\mathbf{A};Q_\ell(x_0)) \geq \left(\frac{(1-\delta)R^2}{b\ell^2}\right)g(b) - r(\kappa)\,, \tag{2.4.25}$$

with b defined in (2.4.12). By choosing $\delta = \ell$ and using that $\overline{Q_\ell(x_0)} \subset \{|B_0| > \epsilon\}$, we get,

$$r(\kappa) = \mathcal{O}\left(\ell^3\kappa^2 + \ell^{2\alpha-1} + \frac{1}{\epsilon}\Big((\ell\kappa)^{-1} + \ell\Big)\right)\kappa^2 . \tag{2.4.26}$$

This implies that,

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0)) \geq g\left(\frac{H}{\kappa}\overline{B}_{Q_\ell(x_0)}\right)\kappa^2 - C\left(\ell^3\kappa^2 + \ell^{2\alpha-1} + (\ell\kappa\epsilon)^{-1} + \ell\epsilon^{-1}\right)\kappa^2 .$$

Similarly, in Case 2, according to Remark 2.2.1, we get that,

$$G_{b,Q_R}^{-1}(v) \geq m_0(b, R) - C_2\left(\kappa\ell + \kappa^2\ell^3\right),$$

and the rest of the proof is as for Case 1. ☐

2.5 Proof of Theorem 2.1.1

2.5.1 Upper bound

Proposition 2.5.1. *There exist positive constants C and κ_0 such that, if* (2.3.10) *holds, then the ground state energy* $\mathrm{E_g}(\kappa, H)$ *in* (2.1.6) *satisfies,*

$$\mathrm{E_g}(\kappa, H) \leq \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + C\kappa^{\frac{15}{8}} .$$

Proof. Let $\ell = \ell(\kappa)$ and $\epsilon = \epsilon(\kappa)$ be positive parameters such that $\kappa^{-1} \ll \ell \ll 1$ and $\kappa^{-1} \ll \epsilon \ll 1$ as $\kappa \to \infty$. For some $\beta \in (0,1)$, $\mu \in (0,1)$ to be determined later, we will choose

$$\ell = \kappa^{-\beta}, \quad \epsilon = \kappa^{-\mu} . \tag{2.5.1}$$

Consider the lattice $\Gamma_\ell := \ell\mathbb{Z} \times \ell\mathbb{Z}$ and write for $\gamma \in \Gamma_\ell$, $Q_{\gamma,\ell} = Q_\ell(\gamma)$. For any $\gamma \in \Gamma_\ell$ such that $\overline{Q_{\gamma,\ell}} \subset \Omega \cap \{|B_0| > \epsilon\}$ let

$$\underline{B}_{\gamma,\ell} = \inf_{x \in Q_{\gamma,\ell}} |B_0(x)| . \tag{2.5.2}$$

Let

$$\mathcal{I}_{\ell,\epsilon} = \left\{\gamma \ : \ \overline{Q_{\gamma,\ell}} \subset \Omega \cap \{|B_0| > \epsilon\}\right\},$$

$$N = \operatorname{Card} \mathcal{I}_{\ell,\epsilon} ,$$

and

$$\Omega_{\ell,\epsilon} = \operatorname{int}\left(\cup_{\gamma \in \mathcal{I}_{\ell,\epsilon}} \overline{Q_{\gamma,\ell}}\right) .$$

It follows from (2.1.2) that :

$$N = |\Omega|\ell^{-2} + \mathcal{O}(\epsilon\ell^{-2}) + \mathcal{O}(\ell^{-1}) \text{ as } \ell \to 0 \text{ and } \epsilon \to 0 .$$

Let

$$b=\frac{H}{\kappa}\underline{B}_{\gamma,\ell}\,,\qquad R=\ell\sqrt{\kappa H\underline{B}_{\gamma,\ell}}\,,\tag{2.5.3}$$

and u_R be a minimizer of the functional in (2.2.1), i.e.

$$m_0(b,R)=\int_{Q_R}\left(b|(\nabla-i\mathbf{A}_0)u_R|^2-|u_R|^2+\frac{1}{2}|u_R|^4\right)\,dx\,.$$

We will need the function φ_γ introduced in Lemma 2.8.3 which satisfies

$$|\mathbf{F}(x)-\sigma_{\gamma,\ell}\,\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma)-\nabla\varphi_\gamma(x)|\le C\ell^2\,,\ \text{ in } \mathrm{Q}_{\gamma,\ell}\,,$$

where $\sigma_{\gamma,\ell}$ is the sign of B_0 in $Q_{\gamma,\ell}$.
We define the function,

$$v(x)=\begin{cases} e^{-i\kappa H\varphi_\gamma}u_R\Big(\dfrac{R}{\ell}(x-\gamma)\Big) & \text{if } x\in Q_{\gamma,\ell}\subset\{B_0>\epsilon\}\\ e^{-i\kappa H\varphi_\gamma}\overline{u_R}\Big(\dfrac{R}{\ell}(x-\gamma)\Big) & \text{if } x\in Q_{\gamma,\ell}\subset\{B_0<-\epsilon\}\\ 0 & \text{if } x\in\Omega\setminus\Omega_{\ell,\epsilon}\end{cases}\,.$$

Since $u_R\in H^1_0(Q_R)$, then $v\in H^1(\Omega)$. We compute the energy of the configuration $(v,\mathbf{F})$. We get,

$$\begin{aligned}\mathcal{E}(v,\mathbf{F})&=\int_\Omega\left(|(\nabla-i\kappa H\mathbf{F})v|^2-\kappa^2|v|^2+\frac{\kappa^2}{2}|v|^4\right)\,dx\\ &=\sum_{\gamma\in\mathcal{I}_{\ell,\epsilon}}\mathcal{E}_0(v,\mathbf{F};Q_{\gamma,\ell})\,.\end{aligned}\tag{2.5.4}$$

We estimate the term $\mathcal{E}_0(v,\mathbf{F};Q_{\gamma,\ell})$ from above and we write :

$$\begin{aligned}\mathcal{E}_0(v,\mathbf{F};Q_{\gamma,\ell})&=\int_{Q_{\gamma,\ell}}|(\nabla-i\kappa H\mathbf{F})v|^2-\kappa^2|v|^2+\frac{\kappa^2}{2}|v|^4\,dx\\ &=\int_{Q_{\gamma,\ell}}\Big|\Big(\nabla-i\kappa H\big(\sigma_{\gamma,\ell}\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma)+\nabla\varphi_\gamma(x)\big)\Big)v\\ &\qquad\qquad-i\kappa H\big(\mathbf{F}-\sigma_{\gamma,\ell}\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma)-\nabla\varphi_\gamma(x)\big)\Big)v\Big|^2-\kappa^2|v|^2+\frac{\kappa^2}{2}|v|^4\,dx\\ &\le\int_{Q_{\gamma,\ell}}(1+\delta)\Big|\Big(\nabla-i\kappa H\big(\sigma_{\gamma,\ell}\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma)+\nabla\varphi_\gamma(x)\big)\Big)v\Big|^2-\kappa^2|v|^2+\frac{\kappa^2}{2}|v|^4dx\\ &\qquad+C(1+\delta^{-1})(\kappa H)^2\int_{Q_{\gamma,\ell}}\Big|\Big(\mathbf{F}-\sigma_{\gamma,\ell}\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma)-\nabla\varphi_\gamma(x)\Big)v\Big|^2\,dx\\ &\le(1+\delta)\mathcal{E}_0(e^{-i\kappa H\varphi_\gamma}v,\sigma_{\gamma,\ell}\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma);Q_{\gamma,\ell})+C(\delta\kappa^2+\delta^{-1}\kappa^4\ell^4)\int_{Q_{\gamma,\ell}}|v|^2d\,x\,.\end{aligned}\tag{2.5.5}$$

Having in mind that u_R is a minimizer of the functional in (2.2.1), and using the estimate in

(2.3.1) we get :

$$\int_{Q_{\gamma,\ell}} |v|^2 dx \leq |Q_{\gamma,\ell}|\,.$$

Remark 2.2.1 and a change of variables give us,

$$\int_{Q_{\gamma,\ell}} \left(|(\nabla - i\kappa H \sigma_{\gamma,\ell}(\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma))e^{-i\kappa H\varphi_\gamma} v|^2 - \kappa^2|v|^2 + \frac{\kappa^2}{2}|v|^4 \right) dx = \frac{m_0(b,R)}{b}\,.$$

We insert this into (2.5.5) to obtain,

$$\mathcal{E}_0(v,\mathbf{F};Q_{\gamma,\ell}) \leq (1+\delta)\frac{m_0(b,R)}{b} + C(\delta\kappa^2 + \delta^{-1}\kappa^4\ell^4)\ell^2\,. \tag{2.5.6}$$

We know from Theorem 2.2.2 that $m_0(b,R) \leq g(b)R^2 + CR$ for all $b \in [0,1]$ and R sufficiently large, where b introduced in (2.5.3). We choose $\delta = \ell$ in (2.5.6). That way we get,

$$\mathcal{E}_0(v,\mathbf{F};Q_{\gamma,\ell}) \leq g\left(\frac{H}{\kappa}\underline{B}_{\gamma,\ell}\right)\ell^2\kappa^2 + C\left(\frac{1}{\kappa\ell\sqrt{\epsilon}} + \ell + \kappa^2\ell^3\right)\ell^2\kappa^2\,. \tag{2.5.7}$$

Summing (2.5.7) over γ in $I_{\ell,\epsilon}$, we recognize the lower Riemann sum of $x \to g\left(\frac{H}{\kappa}|B_0(x)|\right)$. By monotonicity of g, g is Riemann-integrable and its integral is larger than any lower Riemann sum. Thus :

$$\mathcal{E}(v,\mathbf{F}) \leq \left(\int_{\Omega_{\ell,\epsilon}} g\left(\frac{H}{\kappa}|B_0(x)|\right) dx\right)\kappa^2 + C\left(\frac{1}{\kappa\ell\sqrt{\epsilon}} + \ell + \kappa^2\ell^3\right)\kappa^2\,. \tag{2.5.8}$$

Notice that using the regularity of $\partial\Omega$ and (2.1.2), there exists $C > 0$ such that :

$$|\Omega \setminus \Omega_{\ell,\epsilon}| = \mathcal{O}\left(\ell|\partial\Omega| + C\epsilon\right)\,, \tag{2.5.9}$$

as ϵ and ℓ tend to 0.
Thus, we get by using the properties of g in Theorem 2.2.2,

$$\int_{\Omega_{\ell,\epsilon}} g\left(\frac{H}{\kappa}|B_0(x)|\right) dx \leq \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + \frac{1}{2}|\Omega \setminus \Omega_{\ell,\epsilon}|.$$

This implies that,

$$\mathcal{E}(v,\mathbf{F}) \leq \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + C\left(\frac{1}{\kappa\ell\sqrt{\epsilon}} + \ell + \epsilon + \kappa^2\ell^3\right)\kappa^2\,. \tag{2.5.10}$$

We choose in (2.5.1)

$$\beta = \frac{3}{4} \text{ and } \mu = \frac{1}{8}. \tag{2.5.11}$$

With this choice, we infer from (2.5.10),

$$\mathcal{E}(v,\mathbf{F}) \leq \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + C_1\kappa^{\frac{15}{8}}\,, \tag{2.5.12}$$

and

$$\ell^2\kappa^2\epsilon = \kappa^{\frac{3}{8}} > 1\,. \tag{2.5.13}$$

This finishes the proof of Proposition 2.5.1. □

Remark 2.5.2. In the case when B_0 does not vanish in Ω, ϵ disappears and $\{x \in \Omega; |B_0(x)| > 0\} = \Omega$. Consequently, the Ginzburg-Lundau energy of $(v, \mathbf{F})$ in (2.4.2) satisfies :

$$\mathcal{E}(v,\mathbf{F}) \leq \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx + C\left(\frac{1}{\kappa\ell} + \ell + \kappa^2\ell^3\right)\kappa^2\,.$$

We take the same choice of β as in (2.5.11), then the ground state energy $\mathrm{E_g}(\kappa, H)$ in (2.1.6) satisfies,

$$\mathrm{E_g}(\kappa,H) \leq \kappa^2\int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx + C\kappa^{\frac{7}{4}}\,.$$

2.5.2 Lower bound

We now establish a lower bound for the ground state energy $\mathrm{E_g}(\kappa, H)$ in (2.1.6). The parameters ϵ and ℓ have the same form as in (2.5.1).
Let

$$\overline{B}_{\gamma,\ell} = \sup_{x\in Q_{\gamma,\ell}} |B_0(x)|\,, \tag{2.5.14}$$

and

$$b_{\gamma,\ell} = \frac{H}{\kappa}\overline{B}_{\gamma,\ell}\,,\; R = \ell\sqrt{\kappa H\overline{B}_{\gamma,\ell}}\,, \tag{2.5.15}$$

If $(\psi, \mathbf{A})$ is a minimizer of (2.1.1), we have,

$$\mathrm{E_g}(\kappa,H) = \mathcal{E}_0(\psi,\mathbf{A};\Omega_{\ell,\epsilon}) + \mathcal{E}_0(\psi,\mathbf{A};\Omega\setminus\Omega_{\ell,\epsilon}) + (\kappa H)^2\int_\Omega|\operatorname{curl}(\mathbf{A}-\mathbf{F})|^2\,dx\,, \tag{2.5.16}$$

where, for any $D \subset \Omega$, the energy $\mathcal{E}_0(\psi, \mathbf{A}; D)$ is introduced in (2.4.1). Since the magnetic energy term is positive, we may write,

$$\mathrm{E_g}(\kappa,H) \geq \mathcal{E}_0(\psi,\mathbf{A};\Omega_{\ell,\epsilon}) + \mathcal{E}_0(\psi,\mathbf{A};\Omega\setminus\Omega_{\ell,\epsilon})\,. \tag{2.5.17}$$

Thus, we get by using (2.3.1), (2.3.11), and (2.5.9) :

$$\begin{aligned}|\mathcal{E}_0(\psi,\mathbf{A};\Omega\setminus\Omega_{\ell,\epsilon})| &\leq \int_{\Omega\setminus\Omega_{\ell,\epsilon}}|(\nabla - i\kappa H\mathbf{A})\psi|^2 + \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\,dx\\ &\leq |\Omega\setminus\Omega_{\ell,\epsilon}|\left(C_1\kappa^2\|\psi\|^2_{L^\infty(\Omega)} + \kappa^2\|\psi\|^2_{L^\infty(\Omega)} + \frac{\kappa^2}{2}\|\psi\|^4_{L^\infty(\Omega)}\right)\\ &\leq C_2(\ell+\epsilon)\kappa^2\,.\end{aligned} \tag{2.5.18}$$

To estimate $\mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon})$, we notice that,

$$\mathcal{E}_0(\psi,\mathbf{A};\Omega_{\ell,\epsilon}) = \sum_{\gamma\in\mathcal{I}_{\ell,\epsilon}}\mathcal{E}_0(\psi,\mathbf{A};Q_{\gamma,\ell})\,.$$

Using Proposition 2.4.2 with $\alpha = \frac{2}{3}$ and (2.5.18) with $\beta = \frac{3}{4}$ and $\mu = \frac{1}{8}$ in (2.5.1), we get,

$$\begin{aligned}\mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon}) &\geq \sum_{\gamma \in \mathcal{I}_{\ell,\epsilon}} g\left(\frac{H}{\kappa}\overline{B}_{Q_\ell(x_0)}\right)\ell^2\kappa^2 - C\left(\ell^3\kappa^2 + \ell^{2\alpha-1} + (\ell\kappa\epsilon)^{-1} + \ell\epsilon^{-1}\right)\kappa^2\\ &\geq \kappa^2 \sum_{\gamma \in \mathcal{I}_{\ell,\epsilon}} g\left(\frac{H}{\kappa}\overline{B}_{Q_\ell(x_0)}\right)\ell^2 - C_1\kappa^{\frac{15}{8}},\end{aligned}$$

and

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega \setminus \Omega_{\ell,\epsilon}) \geq -C_2\kappa^{\frac{15}{8}}\,. \tag{2.5.19}$$

As for the upper bound, we can use the monotonicity of g and recognize that the sum above is an upper Riemann sum of g. In that way, we get,

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon}) \geq \kappa^2 \int_{\Omega_{\ell,\epsilon}} g\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx - C_1\kappa^{\frac{15}{8}}\,.$$

Notice that $\Omega_{\ell,\epsilon} \subset \Omega$ and that $g \leq 0$, we deduce that,

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon}) \geq \kappa^2 \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx - C_1\kappa^{\frac{15}{8}}\,. \tag{2.5.20}$$

Finally, putting (2.5.19) and (2.5.20) into (2.5.17), we obtain

$$\mathrm{E_g}(\kappa, H) \geq \kappa^2 \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx - C\kappa^{\frac{15}{8}}\,. \tag{2.5.21}$$

Remark 2.5.3. When B_0 does not vanish, the local energy in $Q_\ell(x_0)$ in Proposition 2.4.2 becomes :

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0)) \geq g\left(\frac{H}{\kappa}\overline{B}_{Q_\ell(x_0)}\right)\kappa^2 - C\left(\ell^3\kappa^2 + \ell^{2\alpha-1} + (\ell\kappa)^{-1}\right)\kappa^2\,.$$

Similarly, we choose $\alpha = \frac{2}{3}$ and $\ell = \kappa^{-\frac{3}{4}}$, we get

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega \setminus \Omega_{\ell,\epsilon}) \geq -C_2\kappa^{\frac{7}{4}}\,, \tag{2.5.22}$$

and

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon}) \geq \kappa^2 \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx - C_1\kappa^{\frac{7}{4}}\,. \tag{2.5.23}$$

As a consequence of (2.5.22) and (2.5.23), (2.5.21) becomes

$$\mathrm{E_g}(\kappa, H) \geq \kappa^2 \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx - C\kappa^{\frac{7}{4}}\,. \tag{2.5.24}$$

2.5.3 Proof of Corollary 2.1.2

If $(\psi, \mathbf{A})$ is a minimizer of (2.1.1), we have,

$$\mathcal{E}(\psi, \mathbf{A}; \Omega) = \mathcal{E}_0(\psi, \mathbf{A}; \Omega) + (\kappa H)^2 \int_\Omega |\operatorname{curl}(\mathbf{A} - \mathbf{F})|^2 \, dx. \tag{2.5.25}$$

Theorem 2.1.1 tells us that

$$\mathcal{E}(\psi, \mathbf{A}; \Omega) \leq \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + C\kappa^{\tau_0} .$$

This implies that

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega) + (\kappa H)^2 \int_\Omega |\operatorname{curl}(\mathbf{A} - \mathbf{F})|^2 \, dx \leq \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + C_1\kappa^{\tau_0} . \tag{2.5.26}$$

Using (2.5.19), (2.5.20), (2.5.22) and (2.5.23), we get

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega) \geq \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx - C_2\kappa^{\tau_0} . \tag{2.5.27}$$

Putting (2.5.27) into (2.5.26), we get

$$\begin{aligned} - C_2\kappa^{\tau_0} + \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + (\kappa H)^2 \int_\Omega |\operatorname{curl}(\mathbf{A} - \mathbf{F})|^2 \, dx \leq \\ \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + C_1\kappa^{\tau_0} . \end{aligned} \tag{2.5.28}$$

By simplification, we obtain

$$(\kappa H)^2 \int_\Omega |\operatorname{curl}(\mathbf{A} - \mathbf{F})|^2 \, dx \leq C'\kappa^{\tau_0} . \tag{2.5.29}$$

2.6 Local Energy Estimates

The object of this section is to give an estimates to the Ginzburg-Landau energy (2.4.2) in the open set $D \subset \Omega$.

2.6.1 Main statements

Theorem 2.6.1. *There exist positive constants κ_0 such that if* (2.3.10) *is true and $D \subset \Omega$ is an open set, then the local energy of the minimizer satisfies,*

$$\left|\mathcal{E}(\psi, \mathbf{A}; D) - \kappa^2 \int_D g\left(\frac{H}{\kappa}\,|B_0(x)|\right) dx\right| = o(\kappa^2) . \tag{2.6.1}$$

For all (ℓ, x_0) such that $\overline{Q_\ell(x_0)} \subset \Omega \cap \{|B_0| > \epsilon\}$, we define

$$\underline{B}_{Q_\ell(x_0)} = \inf_{x \in Q_\ell(x_0)} |B_0(x)| \, , \tag{2.6.2}$$

where B_0 is introduced in (2.1.2).

Proposition 2.6.2. *For all $\alpha \in (0,1)$, there exist positive constants C, ϵ_0 and κ_0 such that if* (2.3.10) *is true, $\kappa \geq \kappa_0$, $\ell \in (0, \frac{1}{2})$, $\epsilon \in (0, \epsilon_0)$, $\ell^2\kappa^2\epsilon > 1$, $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\rm div}(\Omega)$ is a minimizer of* (2.1.1)*, and $\overline{Q_\ell(x_0)} \subset \Omega \cap \{|B_0| > \epsilon\}$, then,*

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0)) \leq g\left(\frac{H}{\kappa}\underline{B}_{Q_\ell(x_0)}\right)\kappa^2 + C\left(\ell^3\kappa^2 + \ell^{2\alpha-1} + (\ell\kappa\sqrt{\epsilon})^{-1}\right)\kappa^2 .$$

Here $g(\cdot)$ is the function introduced in (2.2.5) *and $\mathcal{E}_0$ is the functional in* (2.4.1).

Proof. As explained earlier in the proof of Lemma 2.4.1 in (2.4.5), we may suppose after performing a gauge transformation that the magnetic potential $\mathbf{A}$ satisfies,

$$|\mathbf{A}(x) - \mathbf{F}(x)| \leq C\frac{\ell^\alpha}{H}, \quad \forall\, x \in Q_\ell(x_0) . \tag{2.6.3}$$

Let

$$b = \frac{H}{\kappa}\underline{B}_{Q_\ell(x_0)}, \;\; R = \ell\sqrt{\kappa H \underline{B}_{Q_\ell(x_0)}} , \tag{2.6.4}$$

and $u_R \in H^1_0(Q_R)$ be the minimizer of the functional G^{+1}_{b,Q_R} introduced in (2.2.1). Let $\chi_R \in C_c^\infty(\mathbb{R}^2)$ be a cut-off function such that,

$$0 \leq \chi_R \leq 1 \quad \text{in } \mathbb{R}^2 , \quad \operatorname{supp}\chi_R \subset Q_{R+1} , \quad \chi_R = 1 \quad \text{in } Q_R ,$$

and $|\nabla\chi_R| \leq C$ for some universal constant C.
Let $\eta_R(x) = 1 - \chi_R\left(\frac{R}{\ell}(x - x_0)\right)$ for all $x \in \mathbb{R}^2$ and $\widetilde{\ell} = \ell\left(1 + \frac{1}{R}\right)$.
This implies that,

$$\eta_R(x) = 0 \qquad \text{in} \quad Q_\ell(x_0) \tag{2.6.5}$$

$$0 \leq \eta_R(x) \leq 1 \qquad \text{in} \quad Q_{\widetilde{\ell}}(x_0) \setminus Q_\ell(x_0) \tag{2.6.6}$$

$$\eta_R(x) = 1 \qquad \text{in} \quad \Omega \setminus Q_{\widetilde{\ell}}(x_0) . \tag{2.6.7}$$

Consider the function $w(x)$ defined as follows,

$$w(x) = \eta_R(x)\psi(x) \qquad \text{in} \;\; \Omega \setminus Q_\ell(x_0),$$

and, if $x \in Q_\ell(x_0)$,

$$w(x) = \begin{cases} e^{i\kappa H\varphi}u_R\left(\frac{R}{\ell}(x - x_0)\right) & \text{if} \;\; Q_\ell(x_0) \subset \{B_0 > \epsilon\} \cap \Omega \\ e^{i\kappa H\varphi}\overline{u}_R\left(\frac{R}{\ell}(x - x_0)\right) & \text{if} \;\; Q_\ell(x_0) \subset \{B_0 < -\epsilon\} \cap \Omega . \end{cases}$$

Notice that by construction, $w = \psi$ in $\Omega \setminus Q_{\widetilde{\ell}}(x_0)$. We will prove that, for any $\delta \in (0,1)$ and $\alpha \in (0,1)$,

$$\mathcal{E}(w, \mathbf{A}; \Omega) \leq \mathcal{E}(\psi, \mathbf{A}; \Omega \setminus Q_\ell(x_0)) + (1+\delta)\frac{\ell}{bR}m_0(b,R) + r_0(\kappa)\ell^2 , \tag{2.6.8}$$

and for some constant C, $r_0(\kappa)$ is given as follows,

$$r_0(\kappa) = C\left(\delta + \delta^{-1}\ell^4\kappa^2 + \delta^{-1}\ell^{2\alpha} + \frac{1}{\ell\kappa\sqrt{\epsilon}}\right)\kappa^2\,. \tag{2.6.9}$$

Proof of (2.6.8) **:** With $\mathcal{E}_0$ defined in (2.4.1), we write,

$$\mathcal{E}_0(w,\mathbf{A};\Omega) = \mathcal{E}_1 + \mathcal{E}_2\,, \tag{2.6.10}$$

where

$$\mathcal{E}_1 = \mathcal{E}_0(w,\mathbf{A};\Omega\setminus Q_\ell(x_0))\,,\quad \mathcal{E}_2 = \mathcal{E}_0(w,\mathbf{A};Q_\ell(x_0))\,. \tag{2.6.11}$$

We estimate $\mathcal{E}_1$ and $\mathcal{E}_2$ from above. Starting with $\mathcal{E}_1$ and using (2.6.7), we get,

$$\begin{aligned}
\mathcal{E}_1 &= \int_{\Omega\setminus Q_\ell(x_0)} |(\nabla - i\kappa H\mathbf{A})\eta_R\psi|^2 - \kappa^2|\eta_R\psi|^2 + \frac{\kappa^2}{2}|\eta_R\psi|^4\,dx\\
&= \int_{\Omega\setminus Q_\ell(x_0)} \eta_R^2|(\nabla - i\kappa H\mathbf{A})\psi|^2 + |\nabla\eta_R\psi|^2 + 2R\langle\eta_R(\nabla - i\kappa H\mathbf{A})\psi\,,\,\nabla\eta_R\psi\rangle\\
&\qquad\qquad - \kappa^2\eta_R^2|\psi|^2 + \frac{\kappa^2}{2}\eta_R^4|\psi|^4\,dx\\
&= \mathcal{E}_0(\psi,\mathbf{A};\Omega\setminus Q_\ell(x_0)) + \mathcal{R}(\psi,\mathbf{A})\,,
\end{aligned} \tag{2.6.12}$$

where

$$\begin{aligned}
\mathcal{R}(\psi,\mathbf{A}) = \int_{Q_{\widetilde{\ell}}(x_0)\setminus Q_\ell(x_0)} \Big(& (\eta_R^2 - 1)\left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2\right) + |\psi\nabla\eta_R|^2 + \frac{\kappa^2}{2}(\eta_R^4 - 1)|\psi|^4\\
& + 2\Re\langle\eta_R(\nabla - i\kappa H\mathbf{A})\psi, \psi\nabla\eta_R\rangle\Big)\,dx\,.
\end{aligned}$$

Noticing that $\left|Q_{\widetilde{\ell}}(x_0)\setminus Q_\ell(x_0)\right| \leq \frac{\ell}{\sqrt{\kappa H \underline{B}_{Q_\ell(x_0)}}}$ and using (2.6.6) together with the estimates in (2.3.1), (2.3.10), (2.3.11) and $|\nabla\eta_R| \leq C\dfrac{R}{\ell}$, we get,

$$|\mathcal{R}(\psi,\mathbf{A})| \leq C\frac{\ell\kappa}{\sqrt{\epsilon}}\,. \tag{2.6.13}$$

Inserting (2.6.13) in (2.6.12), we get the following estimate,

$$\mathcal{E}_1 \leq \mathcal{E}_0(\psi,\mathbf{A};\Omega\setminus Q_\ell(x_0)) + C\frac{\ell\kappa}{\sqrt{\epsilon}}\,. \tag{2.6.14}$$

We estimate the term $\mathcal{E}_2$ in (2.6.11). We will need the function φ_0 introduced in Lemma 2.8.3 and satisfying $|\mathbf{F}(x) - \sigma_\ell\,\underline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0) - \nabla\varphi_0(x)| \leq C\ell^2$ in $Q_\ell(x_0)$, where σ_ℓ denotes the sign of B_0. We start with the kinetic energy term and write for any $\delta \in (0,1)$:

$$\begin{aligned}
\mathcal{E}_2 = \int_{Q_\ell(x_0)} \Big| \Big(\nabla - i\kappa H(\sigma_\ell \underline{B}_{Q_\ell(x_0)} \mathbf{A}_0(x-x_0) + \nabla\varphi(x))\Big) w \\
- i\kappa H\Big(\mathbf{A} - \big(\sigma_\ell \underline{B}_{Q_\ell(x_0)} \mathbf{A}_0(x-x_0) + \nabla\varphi(x)\big)\Big)\Big|^2 + \left(-\kappa^2|w|^2 + \frac{\kappa^2}{2}|w|^4\right) dx \\
\leq \int_{Q_\ell(x_0)} (1+\delta)\Big|\Big(\nabla - i\kappa H(\sigma_\ell \underline{B}_{Q_\ell(x_0)} \mathbf{A}_0(x-x_0) + \nabla\varphi(x))\Big)w\Big|^2 - \kappa^2|w|^2 + \frac{\kappa^2}{2}|w|^4\, dx \\
+ (1+\delta^{-1})(\kappa H)^2 \int_{Q_\ell(x_0)} \Big|(\mathbf{A} - \nabla\phi_{x_0} - \mathbf{F})w + (\mathbf{F} - \sigma_\ell \underline{B}_{Q_\ell(x_0)} \mathbf{A}_0(x-x_0) - \nabla\varphi_0(x))w\Big|^2 dx\,.
\end{aligned} \tag{2.6.15}$$

Using the estimate in (2.6.3) together with (2.3.10) and (2.3.1), we deduce the upper bound,

$$\mathcal{E}_2 \leq (1+\delta)\mathcal{E}_0(e^{-i\kappa H\varphi}w, \sigma_\ell \underline{B}_{Q_\ell(x_0)} \mathbf{A}_0(x-x_0); Q_\ell(x_0)) + C(\delta^{-1}\ell^{2\alpha} + \delta^{-1}\ell^4\kappa^2 + \delta)\kappa^2\ell^2\,, \tag{2.6.16}$$

where $\alpha \in (0,1)$.

There are two cases :

Case 1 : If $B_0 > \epsilon$ in $Q_\ell(x_0)$, then $\sigma_\ell = +1$ and

$$w(x) = \begin{cases} e^{i\kappa H\varphi} u_R\left(\frac{R}{\ell}(x-x_0)\right) & \text{in} \quad Q_\ell(x_0) \\ \eta_R(x)\psi(x) & \text{in} \quad \Omega \setminus Q_\ell(x_0)\,. \end{cases}$$

The change of variable $y = \frac{R}{\ell}(x-x_0)$ and (2.4.12) gives us :

$$\begin{aligned}
&\mathcal{E}_0(e^{-i\kappa H\varphi}w, \sigma_\ell \underline{B}_{Q_\ell(x_0)} \mathbf{A}_0(x-x_0); Q_\ell(x_0)) \\
&\qquad = \int_{Q_R} \left(\left|\left(\frac{R}{\ell}\nabla_y - i\frac{R}{\ell}\mathbf{A}_0(y)\right)u_R\right|^2 - \kappa^2|u_R|^2 + \frac{\kappa^2}{2}|u_R|^4\right)\frac{\ell}{R}dy \\
&\qquad = \int_{Q_R} \left(|(\nabla_y - i\mathbf{A}_0(y))u_R|^2 - \frac{\kappa}{H\underline{B}_{Q_\ell(x_0)}}|u_R|^2 + \frac{\kappa}{2H\underline{B}_{Q_\ell(x_0)}}|u_R|^4\right) dy \\
&\qquad = \frac{\kappa}{H\underline{B}_{Q_\ell(x_0)}} \int_{Q_R} b\left(|(\nabla_y - i\mathbf{A}_0(y))u_R|^2 - |u_R|^2 + \frac{1}{2}|u_R|^4\right) dy \\
&\qquad = \frac{1}{b}\, G^{+1}_{b,Q_R}(u_R)\,,
\end{aligned} \tag{2.6.17}$$

where G^{+1}_{b,Q_R} is the functional from (2.2.1).

Case 2 : If $B_0 < -\epsilon$ in $Q_\ell(x_0)$, then $\sigma_\ell = -1$ and

$$w(x) = \begin{cases} e^{i\kappa H\varphi} \overline{u}_R\left(\frac{R}{\ell}(x-x_0)\right) & \text{in} \quad Q_\ell(x_0) \\ \eta_R(x)\psi(x) & \text{in} \quad \Omega \setminus Q_\ell(x_0)\,. \end{cases}$$

Similarly, like in case 1, we have,

$$\mathcal{E}_0(e^{-i\kappa H\varphi}w, \sigma_\ell \underline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0); Q_\ell(x_0)) = \frac{1}{b}\, G^{-1}_{b,Q_R}(\overline{u}_R) = \frac{1}{b}\, G^{+1}_{b,Q_R}(u_R)\,.$$

In both cases we see that,

$$\mathcal{E}_0(e^{-i\kappa H\varphi}w, \sigma_\ell \underline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x-x_0); Q_\ell(x_0)) = \frac{1}{b}\, G^{+1}_{b,Q_R}(u_R) = \frac{m_0(b,R)}{b}. \tag{2.6.18}$$

Inserting (2.6.18) into (2.6.16), we get,

$$\mathcal{E}_2 \leq (1+\delta)\frac{1}{b}m_0(b,R) + C(\delta + \delta^{-1}\ell^4\kappa^2 + \delta^{-1}\ell^{2\alpha})\kappa^2\ell^2\,. \tag{2.6.19}$$

Inserting (2.6.14) and (2.6.19) into (2.6.10), we deduce that,

$$\mathcal{E}_0(w,\mathbf{A}) \leq \mathcal{E}_0(\psi,\mathbf{A};\Omega\setminus Q_\ell(x_0)) + (1+\delta)\frac{1}{b}m_0(b,R) + C(\delta+\delta^{-1}\ell^4\kappa^2+\delta^{-1}\ell^{2\alpha}\kappa^2+(\ell\kappa\sqrt{\epsilon})^{-1})\ell^2\kappa^2\,. \tag{2.6.20}$$

This proves (2.6.8). Now, we show how (2.6.8) proves Proposition 2.6.2. By definition of the minimizer $(\psi,\mathbf{A})$, we have,

$$\mathcal{E}(\psi,\mathbf{A}) \leq \mathcal{E}(w,\mathbf{A};\Omega)\,.$$

Since $\mathcal{E}(\psi,\mathbf{A};\Omega) = \mathcal{E}(\psi,\mathbf{A};\Omega\setminus Q_\ell(x_0)) + \mathcal{E}_0(\psi,\mathbf{A};Q_\ell(x_0))$, the estimate (2.6.8) gives us,

$$\mathcal{E}_0(\psi,\mathbf{A};Q_\ell(x_0)) \leq \frac{(1+\delta)}{b}m_0(b,R) + r_0(\kappa)\,,$$

where $r_0(\kappa)$ is defined in (2.6.9).
Dividing both sides by $|Q_\ell(x_0)| = \ell^2$, we get,

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi,\mathbf{A},Q_\ell(x_0)) \leq \frac{(1+\delta)}{b\ell^2}m_0(b,R) + C\left(\delta+\delta^{-1}\ell^4\kappa^2 + \frac{1}{\ell\kappa\sqrt{\epsilon}} + \delta^{-1}\ell^{2\alpha}\right)\kappa^2\,. \tag{2.6.21}$$

The inequality in (2.2.7) tell us that $m_0(b,R) \leq R^2 g(b) + CR$ for all $b\in[0,1]$ and R sufficiently large. We substitute this into (2.6.21) and we select $\delta=\ell$, so that

$$r_0(\kappa) = \kappa^2\mathcal{O}\left((\ell\kappa\sqrt{\epsilon})^{-1} + \ell^3\kappa^2 + \ell^{2\alpha-1}\right)\,.$$

Using (2.4.12) we get,

$$\begin{aligned}\frac{1}{|Q_\ell(x_0)|}\mathcal{E}(\psi,\mathbf{A},Q_\ell(x_0)) &\leq \frac{(1+\delta)R^2}{b\ell^2}g(b) + \frac{CR}{b\ell^2} + \kappa^2\mathcal{O}\left((\ell\kappa\sqrt{\epsilon})^{-1}+\ell^3\kappa^2+\ell^{2\alpha-1}\right)\\ &\leq g\left(\frac{H}{\kappa}\underline{B}_{Q_\ell(x_0)}\right)\kappa^2 + C\left((\ell\kappa\sqrt{\epsilon})^{-1}+\ell^3\kappa^2+\ell^{2\alpha-1}\right)\kappa^2\,.\end{aligned}$$

This establishes the result of Proposition 2.6.2. □

2.6.2 Proof of Theorem 2.6.1, upper bound

The parameters ℓ and ϵ have the same form as in (2.5.1) and we take the same choice of β and μ as in (2.5.11). Consider the lattice $\Gamma_\ell := \ell\mathbb{Z} \times \ell\mathbb{Z}$ and write, for $\gamma \in \Gamma_\ell$, $Q_{\gamma,\ell} = Q_\ell(\gamma)$. For any $\gamma \in \Gamma_\ell$ such that $\overline{Q_\ell(\gamma)} \subset \Omega \cap \{|B_0| > \epsilon\}$, let :

$$\mathcal{I}_{\ell,\epsilon}(D) = \{\gamma \ : \ \overline{Q_{\gamma,\ell}} \subset D \cap \{|B_0| > \epsilon\}\}, \quad N = \operatorname{Card} \mathcal{I}_{\ell,\epsilon}(D),$$

and

$$D_{\ell,\epsilon} = \operatorname{int}\left(\cup_{\gamma \in \mathcal{I}_{\ell,\epsilon}(D)} \overline{Q_{\gamma,\ell}}\right).$$

Notice that, by (2.1.2),

$$N = |D|\ell^{-2} + \mathcal{O}(\epsilon\ell^{-2}) + \mathcal{O}(\ell^{-1}) \text{ as } \ell \to 0 \text{ and } \epsilon \to 0\,.$$

If $(\psi, \mathbf{A})$ is a minimizer of (2.1.1), we have,

$$\mathcal{E}(\psi, \mathbf{A}; D) = \mathcal{E}_0(\psi, \mathbf{A}; D_{\ell,\epsilon}) + \mathcal{E}_0(\psi, \mathbf{A}; D \setminus D_{\ell,\epsilon}) + (\kappa H)^2 \int_\Omega |\operatorname{curl}\big(\mathbf{A} - \mathbf{F}\big)|^2\, dx\,. \tag{2.6.22}$$

Using Corollary 2.1.2, we may write,

$$\mathcal{E}(\psi, \mathbf{A}; D) \leq \mathcal{E}_0(\psi, \mathbf{A}; D_{\ell,\epsilon}) + \mathcal{E}_0(\psi, \mathbf{A}; D \setminus D_{\ell,\epsilon}) + C\kappa^{\tau_0}\,. \tag{2.6.23}$$

Here $\tau_0 \in (1, 2)$. Notice that

$$|D \setminus D_{\ell,\epsilon}| = \mathcal{O}\left(\ell|\partial D_{\ell,\epsilon}| + \epsilon\right)\,. \tag{2.6.24}$$

We get by using (2.3.1) and (2.3.11) :

$$\begin{aligned}|\mathcal{E}_0(\psi, \mathbf{A}; D \setminus D_{\ell,\epsilon})| &\leq |D \setminus D_{\ell,\epsilon}| \left(C_1\kappa^2\|\psi\|^2_{L^\infty(D)} + \kappa^2\|\psi\|^2_{L^\infty(D)} + \frac{\kappa^2}{2}\|\psi\|^4_{L^\infty(D)}\right)\\ &\leq C_2(\ell + \epsilon)\kappa^2\,.\end{aligned} \tag{2.6.25}$$

To estimate $\mathcal{E}_0(\psi, \mathbf{A}; D_{\ell,\epsilon})$, we notice that,

$$\mathcal{E}_0(\psi, \mathbf{A}; D_{\ell,\epsilon}) = \sum_{\gamma \in \mathcal{I}_{\ell,\epsilon}(D)} \mathcal{E}_0(\psi, \mathbf{A}; Q_{\gamma,\ell})\,.$$

Using Proposition 2.6.2 and the estimates in (2.6.25) with $\beta = \frac{3}{4}$, $\alpha = \frac{2}{3}$ and $\mu = \frac{1}{8}$, we get,

$$\begin{aligned}\mathcal{E}_0(\psi, \mathbf{A}; D) &\leq \sum_{\gamma \in \mathcal{I}_{\ell,\epsilon}(D)} g\left(\frac{H}{\kappa}\underline{B}_{Q_\ell(x_0)}\right)\kappa^2\ell^2 + C\left(\ell^3\kappa^2 + \ell^{2\alpha-1} + (\ell\kappa\sqrt{\epsilon})^{-1} + \epsilon\right)\kappa^2 + C_1\kappa^{\tau_0}\\ &\leq \kappa^2 \sum_{\gamma \in \mathcal{I}_{\ell,\epsilon}(D)} g\left(\frac{H}{\kappa}\underline{B}_{Q_\ell(x_0)}\right)\ell^2 + C_2\kappa^{\tau_0}\,,\end{aligned}$$

where

$$\underline{B}_{Q_\ell(x_0)} = \sup_{x\in Q_\ell(x_0)} B_0(x)\,.$$

Recognizing the lower Riemann sum of $x \mapsto g\left(\frac{H}{\kappa}B_0(x)\right)$, and using the monotonicity of g we get :

$$\mathcal{E}_0(\psi,\mathbf{A};D) \le \kappa^2 \int_{D_{\ell,\epsilon}} g\left(\frac{H}{\kappa}B_0(x)\right)\,dx + C_2\kappa^{\tau_0}\,. \tag{2.6.26}$$

Thus, we get by using (2.6.24) and the property of g in Theorem 2.2.2,

$$\kappa^2 \int_{D_{\ell,\epsilon}} g\left(\frac{H}{\kappa}B_0(x)\right)\,dx \le \kappa^2 \int_{D} g\left(\frac{H}{\kappa}B_0(x)\right)\,dx + C_3\kappa^{\tau_0}\,.$$

This finishes the proof of the upper bound.

2.6.3 Lower bound

We keep the same notation as in the derivation of the upper bound. We start with (2.6.22) and write,

$$\mathcal{E}(\psi,\mathbf{A};D) \ge \mathcal{E}_0(\psi,\mathbf{A};D_{\ell,\epsilon}) + \mathcal{E}_0(\psi,\mathbf{A};D\setminus D_{\ell,\epsilon})\,. \tag{2.6.27}$$

Similarly, as we did for the Lower bound 2.5.2, we get,

$$\mathcal{E}(\psi,\mathbf{A};D) \ge \kappa^2 \int_{D} g\left(\frac{H}{\kappa}B_0(x)\right)\,dx - C\kappa^{\tau_0}\,. \tag{2.6.28}$$

This finish the proof of Theorem 2.6.1.

2.7 Proof of Theorem 2.1.4

2.7.1 Proof of(2.1.9)

Let $(\psi,\mathbf{A})$ be a solution of (2.1.5) and $\tau_1 = \tau_0 - 2$. Then ψ satisfies,

$$-(\nabla - i\kappa H\mathbf{A})^2\psi = \kappa^2(1-|\psi|^2)\psi \quad \text{in} \quad \Omega\,. \tag{2.7.1}$$

We multiply both sides of the equation in (2.7.1) by $\overline{\psi}$ then we integrate over D. An integration by parts gives us,

$$\int_D \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \kappa^2|\psi|^4\right)\,dx - \int_{\partial D} \nu\cdot(\nabla - i\kappa H\mathbf{A})\psi\,\overline{\psi}\,d\sigma(x) = 0\,. \tag{2.7.2}$$

Using the estimates (2.3.1), (2.3.10) and (2.3.11) , we get that the boundary term which is not necessary 0 if $D \neq \Omega$ above is $\mathcal{O}(\kappa)$. So, we rewrite (2.7.2) as follows,

$$-\frac{1}{2}\kappa^2 \int_D |\psi|^4\,dx = \mathcal{E}_0(\psi,\mathbf{A};D) + \mathcal{O}(\kappa)\,. \tag{2.7.3}$$

Using (2.6.28), we conclude that,

$$\frac{1}{2}\int_D |\psi|^4\,dx \leq -\int_D g\left(\frac{H}{\kappa}B_0(x)\right)dx + C\kappa^{\tau_1}. \tag{2.7.4}$$

2.7.2 Proof of (2.1.10)

If $(\psi, \mathbf{A})$ is a minimizer of (2.1.1), then (2.7.3) is still true. We apply in this case Theorem 2.6.1 to write an upper bound of $\mathcal{E}_0(\psi, \mathbf{A}; D)$. Consequently, we deduce that,

$$\frac{1}{2}\int_D |\psi|^4\,dx \geq -\int_D g\left(\frac{H}{\kappa}B_0(x)\right)dx - C\kappa^{\tau_1}. \tag{2.7.5}$$

Combining the upper bound in (2.7.5) with the lower bound in (2.7.4) finishes the proof of Theorem 2.1.4.

2.8 Useful gauge transformation

2.8.1 L^p-regularity for the curl-div system

We consider the two dimensional case. We denote, for $k \in \mathbb{N}$, by $W^{k,p}_{\mathrm{div}}(\Omega)$ the space

$$W^{k,p}_{\mathrm{div}}(\Omega) = \{\mathbf{A} \in W^{k,p}(\Omega), \mathrm{div}\mathbf{A} = 0 \,\mathrm{and}\, \mathbf{A}\cdot\nu = 0 \,\mathrm{on}\, \partial\Omega\}.$$

Then we have the following L^p regularity for the curl-div system.

Proposition 2.8.1. *Let $1 \leq p < \infty$. If $\mathbf{A} \in W^{1,p}_{\mathrm{div}}(\Omega)$ satisfies $\mathrm{curl}\,\mathbf{A} \in W^{k,p}(\Omega)$, for some $k \geq 0$, then $\mathbf{A} \in W^{k+1,p}_{\mathrm{div}}(\Omega)$.*

Proof. If $\mathbf{A}$ belongs to $W^{1,p}_{\mathrm{div}}(\Omega)$ and $\mathrm{curl}\,\mathbf{A} \in L^p(\Omega)$, then there exists $\psi \in W^{2,p}(\Omega)$ such that $\mathbf{A} = (-\partial_{x_2}\psi, \partial_{x_1}\psi)$, $-\Delta\psi = \mathrm{curl}\,\mathbf{A}$, with $\psi = 0$ on $\partial\Omega$. This is simply the Dirichlet L^p problem for the Laplacian (See [15], Section A.1). The result we need for proving the proposition is then that if $-\Delta\psi$ is in addition in $W^{k,p}(\Omega)$ then $\psi \in W^{k+2,p}(\Omega)$. This is simply an L^p regularity result for the Dirichlet problem for the Laplacian which is described in ([15], Section F.4). ☐

2.8.2 Construction of φ_{x_0}.

Lemma 2.8.2. *If $B_0 \in L^2(\Omega)$, then there exists a unique $\mathbf{F} \in H^1_{\mathrm{div}}(\Omega)$ such that,*

$$\mathrm{curl}\,\mathbf{F} = B_0\,. \tag{2.8.1}$$

Proof. The proof is standard, see [25]. Let $\mathbf{F} = \begin{bmatrix} \partial_{x_2} f \\ -\partial_{x_1} f \end{bmatrix}$, where $f \in H^2(\Omega)\bigcap H^1_0(\Omega)$ is the unique solution of

$$-\Delta f = B_0 \quad \text{in } \Omega\,. \tag{2.8.2}$$

Then we deduce from the Dirichlet condition satisfied by f that $\tau \cdot \nabla f = 0 \quad \text{on } \partial\Omega$ which is equivalent to $\nu \cdot \mathbf{F} = 0 \quad \text{on } \partial\Omega$. This finishes the proof of Lemma 2.8.2. $\square$

We continue with a lemma that will be useful in estimating the Ginzburg-Landau functional.

Lemma 2.8.3. *There exists a positive constant C such that, if $\ell \in (0,1)$ and $x_0 \in \Omega$ are such that $\overline{Q_\ell(x_0)} \subset \Omega$, then for any $\widetilde{x_0} \in \overline{Q_\ell(x_0)}$, there exists a function $\varphi_0 \in H^1(\Omega)$ such that the magnetic potential $\mathbf{F}$ satisfies,*

$$|\mathbf{F}(x) - \nabla\varphi_0(x) - B_0(\widetilde{x_0})\mathbf{A}_0(x - x_0)| \le C\ell^2\,, \quad \Big(x \in Q_\ell(x_0)\Big)\,, \tag{2.8.3}$$

where B_0 is the function introduced in (2.1.2) *and $\mathbf{A}_0$ is the magnetic potential introduced in* (2.2.2).

Proof. We use Taylor formula near $\widetilde{x_0}$ to order 2 and get :

$$\mathbf{F}(x) = \mathbf{F}(\widetilde{x_0}) + M(x - \widetilde{x_0}) + \mathcal{O}(|x - \widetilde{x_0}|^2)\,, \qquad \forall x \in Q_\ell(x_0)\,, \tag{2.8.4}$$

where

$$M = D\mathbf{F}(\widetilde{x_0}) = \begin{bmatrix} \dfrac{\partial \mathbf{F}^1}{\partial x_1}|_{\widetilde{x_0}} & \dfrac{\partial \mathbf{F}^1}{\partial x_2}|_{\widetilde{x_0}} \\ \dfrac{\partial \mathbf{F}^2}{\partial x_1}|_{\widetilde{x_0}} & \dfrac{\partial \mathbf{F}^2}{\partial x_2}|_{\widetilde{x_0}} \end{bmatrix}.$$

We can write M as the sum of two matrices, $M = M^s + M^{as}$, where $M^s = \frac{M+M^t}{2}$ is symmetric and $M^{as} = \frac{M-M^t}{2}$ is antisymmetric.
Notice that $\operatorname{curl}\mathbf{F}(\widetilde{x_0}) = \dfrac{\partial \mathbf{F}^2}{\partial x_1}|_{\widetilde{x_0}} - \dfrac{\partial \mathbf{F}^1}{\partial x_2}|_{\widetilde{x_0}} = B_0(\widetilde{x_0})$. Consequently,

$$M^{as} = \begin{bmatrix} 0 & -B_0/2 \\ B_0/2 & 0 \end{bmatrix}.$$

Substitution into M gives as that,

$$M(x - x_0) = \nabla\phi_0(x) + B_0(\widetilde{x_0})\mathbf{A}_0(x - x_0)\,,$$

where $\mathbf{A}_0(x) = \frac{1}{2}(-x_2, x_1)$ and the function ϕ_0 is defined by

$$\phi_0(x) = \frac{1}{2}\left\langle \left(\frac{M + M^t}{2}\right)(x - x_0), (x - x_0)\right\rangle.$$

Let $\varphi_0(x) = \phi_0(x) + (\mathbf{F}(\widetilde{x_0}) + M(x_0 - \widetilde{x_0}))\cdot x$. Substitution into (2.8.4) gives as that,

$$\mathbf{F} = B_0(\widetilde{x_0})\mathbf{A}_0(x - x_0) + \nabla\varphi_0(x) + \mathcal{O}\left(|x - \widetilde{x_0}|^2\right).$$

Notice that, if $x \in Q_\ell(x_0)$, then $|x - \widetilde{x_0}| \le \ell\sqrt{2}$. This finishes the proof of Lemma 2.8.3. $\square$

Remark 2.8.4. We will apply this lemma by considering $\widetilde{x}_0$ such that $B_0(\widetilde{x}_0) = \sup_{Q_\ell(x_0)} B_0(x)$ or $B_0(\widetilde{x}_0) = \inf_{Q_\ell(x_0)} B_0(x)$.

Chapitre 3

Energy and vorticity of the Ginzburg-Landau model with variable magnetic field

We consider the Ginzburg-Landau functional with a variable applied magnetic field in a bounded and smooth two dimensional domain. The applied magnetic field varies smoothly and is allowed to vanish non-degenerately along a curve. Assuming that the strength of the applied magnetic field varies between two characteristic scales, and the Ginzburg-Landau parameter tends to $+\infty$, we determine an accurate asymptotic formula for the minimizing energy and show that the energy minimizers have vortices. The new aspect in the presence of a variable magnetic field is that the density of vortices in the sample is not uniform.

3.1 Introduction

We consider a bounded, open and simply connected set $\Omega \subset \mathbb{R}^2$ with smooth boundary. We suppose that Ω models a superconducting sample subject to an applied external magnetic field. The energy of the sample is given by the Ginzburg-Landau functional,

$$\mathcal{E}_{\kappa,H}(\psi,\mathbf{A}) = \int_\Omega \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 + \frac{\kappa^2}{2}(1-|\psi|^2)^2 \right) dx + \kappa^2 H^2 \int_\Omega |\operatorname{curl}\mathbf{A} - B_0|^2\, dx\,. \quad (3.1.1)$$

Here κ and H are two positive parameters, to simplify we will consider that $H = H(\kappa)$. The wave function (order parameter) $\psi \in H^1(\Omega;\mathbb{C})$ and the magnetic potential $\mathbf{A} \in H^1_{\mathrm{div}}(\Omega)$. The space $H^1_{\mathrm{div}}(\Omega)$ is defined in (3.1.4) below. Finally, the function $B_0 \in C^\infty(\overline{\Omega})$ gives the intensity of the external variable magnetic field. Let $\Gamma = \{x \in \overline{\Omega}, B_0(x) = 0\}$, then, we assume that B_0 satisfies :

$$\begin{cases} |B_0| + |\nabla B_0| > 0 & \text{in } \overline{\Omega} \\ \nabla B_0 \times \vec{n} \neq 0 & \text{on } \Gamma \cap \partial\Omega\,. \end{cases} \quad (3.1.2)$$

The assumption in (3.1.2) implies that for any open set ω relatively compact in Ω the set $\Gamma \cap \omega$ will be either empty, or consists of a union of smooth curves. Here, the definition of the functional (3.1.1) is taken as in [14]. In [13], the scaling for the intensity of the external magnetic field (denoted by h) is different. We choose the scaling from [14] for convenience when estimating the ground state energy of the functional.

Let $\mathbf{F} : \Omega \to \mathbb{R}^2$ be the unique vector field such that,

$$\operatorname{div} \mathbf{F} = 0 \text{ and } \operatorname{curl} \mathbf{F} = B_0 \text{ in } \Omega\,, \quad \nu \cdot \mathbf{F} = 0 \text{ on } \partial\Omega. \tag{3.1.3}$$

The vector ν is the unit interior normal vector of $\partial\Omega$. We define the space,

$$H^1_{\rm div}(\Omega) = \{\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2) \in H^1(\Omega)^2 \ : \ \operatorname{div} \mathbf{A} = 0 \text{ in } \Omega\,,\ \mathbf{A}\cdot\nu = 0 \text{ on} \,\partial\Omega\,\}. \tag{3.1.4}$$

Critical points $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\rm div}(\Omega)$ of $\mathcal{E}_{\kappa,H}$ are weak solutions of the Ginzburg-Landau equations,

$$\begin{cases} -(\nabla - i\kappa H\mathbf{A})^2\psi = \kappa^2(1-|\psi|^2)\psi & \text{in} \quad \Omega \\ -\nabla^{\perp}\operatorname{curl}(\mathbf{A}-\mathbf{F}) = \dfrac{1}{\kappa H}\operatorname{Im}(\overline{\psi}\,(\nabla - i\kappa H\mathbf{A})\psi) & \text{in} \quad \Omega \\ \nu\cdot(\nabla - i\kappa H\mathbf{A})\psi = 0 & \text{on} \quad \partial\Omega \\ \operatorname{curl}\mathbf{A} = \operatorname{curl}\mathbf{F} & \text{on} \quad \partial\Omega\,. \end{cases} \tag{3.1.5}$$

Here, $\operatorname{curl}\mathbf{A} = \partial_{x_1}\mathbf{A}_2 - \partial_{x_2}\mathbf{A}_1$ and $\nabla^{\perp}\operatorname{curl}\mathbf{A} = (\partial_{x_2}(\operatorname{curl}\mathbf{A}), -\partial_{x_1}(\operatorname{curl}\mathbf{A}))$.

For a solution $(\psi, \mathbf{A})$ of (3.1.5), the function ψ describes the superconducting properties of the material and $(\kappa H \operatorname{curl}\mathbf{A})$ is the induced magnetic field. The number κ is a parameter describing the properties of the material, and the number H measures the variation of the intensity of the applied magnetic field. We focus on the regime of large values of κ, $\kappa \to +\infty$.

In this paper, we study the ground state energy defined as follows :

$$\mathrm{E_g}(\kappa, H) = \inf\left\{\mathcal{E}_{\kappa,H}(\psi,\mathbf{A}) \ : \ (\psi,\mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\rm div}(\Omega)\right\}. \tag{3.1.6}$$

More precisely, we give an asymptotic estimate valid when $H(\kappa)$ satisfies :

$$C_{\min}\kappa^{\frac{1}{3}} \le H(\kappa) \ll \kappa \qquad \text{as } \kappa \longrightarrow +\infty\,, \tag{3.1.7}$$

where $C_{\min}$ is a positive constant.
The behavior of $\mathrm{E_g}(\kappa, H)$ involves a function $\hat{f} : [0,1] \longrightarrow [0, \frac{1}{2}]$ introduced in (3.2.10) below. The function $\hat{f}$ is the limit of a simplified Ginzburg-Landau type functional. It has been defined by Sandier-Serfaty in [45], and then analyzed in [3, 18]. This function plays an important role in describing the distribution of superconductivity in the bulk of 2D and 3D samples, see [45], [3, 17, 18] and the recent papers [5, 27].

In Section 3.2, we will state various properties of the function $\hat{f}$. Note for the moment that the function $\hat{f}$ is increasing, continuous and $\hat{f}(b) = \frac{1}{2}$, for all $b \ge 1$.

Under the assumption that $B_0(x)$ satisfies (3.1.2) and that the function $H = H(\kappa)$ satisfies

$$C_1\kappa \le H \le C_2\kappa\,, \tag{3.1.8}$$

where C_1 and C_2 are positive constants, we obtained [1] in [5] that

$$\mathrm{E_g}(\kappa, H) = \kappa^2 \int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx + o\left(\kappa H\right) , \qquad \text{as } \kappa \longrightarrow +\infty . \tag{3.1.9}$$

In this paper, we generalize this result to the case when $H(\kappa)$ satisfies (3.1.7).

Theorem 3.1.1. *Under Assumptions* (3.1.2) *and* (3.1.7)*, the ground state energy in* (3.1.6) *satisfies, as* $\kappa \longrightarrow +\infty$

$$\mathrm{E_g}(\kappa, H) = \kappa^2 \int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx + o\left(\kappa H \ln \frac{\kappa}{H}\right) . \tag{3.1.10}$$

We will see in Remark 3.3.4 that the second term in the right hand side of (3.1.10), which is actually more simply $o(\kappa H \ln \kappa)$ when (3.1.7) is satisfied, is of lower order compared with the leading term. Actually (see in Theorem 3.2.1), the function $\hat{f}$ satisfies

$$\hat{f}(b) = \frac{b}{2} \ln \frac{1}{b} \left(1 + \hat{s}(b)\right) , \qquad \text{as } b \longrightarrow 0 ,$$

with $\hat{s}(b) = o(1)$.
As a consequence of the behaviour of $\hat{f}$ above, (3.1.10) becomes

$$\mathrm{E_g}(\kappa, H) = \frac{1}{2}\kappa H \left[\int_\Omega |B_0(x)| \ln \frac{\kappa}{H|B_0(x)|}\, dx\right] \left(1 + o(1)\right) . \tag{3.1.11}$$

When the magnetic field is constant (i.e B_0 is a constant function), (3.1.11) is proved in [46] under the relaxed condition

$$\frac{\ln \kappa}{\kappa} \ll H \ll \kappa . \tag{3.1.12}$$

The reason why we do not obtain (3.1.11) under the relaxed condition (3.1.12) is probably technical. The method is to construct test configurations with a Dirichlet boundary condition. We can not construct periodic configurations as in [46] because the magnetic field B_0 is variable. The approach used in the proof of Theorem 3.1.1 is close to that in [31] which studies the same problem when $\Omega \subset \mathbb{R}^3$ and B_0 is constant.

Remark 3.1.2. Notice that when $H(\kappa)$ satisfies (3.1.8) we have

$$o(\kappa H) = o\left(\kappa H \left(\left|\ln \frac{H}{\kappa}\right| + 1\right)\right) .$$

If we assume that there exist positive constants $C_{\min}$ and C_1 and $H(\kappa)$ satisfies

$$C_{\min} \kappa^{\frac{1}{3}} \leq H(\kappa) \leq C_1 \kappa , \tag{3.1.13}$$

[1]After a change of notation

then (3.1.9) and (3.1.10) can be rewritten in a unique statement :

$$\mathrm{E_g}(\kappa, H) = \kappa^2 \int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx + o\left(\kappa H\left(\left|\ln \frac{H}{\kappa}\right| + 1\right)\right). \tag{3.1.14}$$

Remark 3.1.3. When the set $\Gamma = \{x \in \overline{\Omega}, B_0(x) = 0\}$ consists of a finite number of smooth curves and the intensity of the magnetic field H satisfies $\kappa \ll H \leq \mathcal{O}(\kappa^2)$, then the energy $\mathrm{E_g}(\kappa, H)$ in (3.1.1) is estimated in [27].

Theorem 3.1.1 admits the following corollary which is useful in the proof of Theorem 3.1.5 below. The content of Corollary 3.1.4 gives us that the magnetic energy is small compared with the leading term in (3.1.14).

Corollary 3.1.4. *Suppose that the assumptions of Theorem 3.1.1 hold. Then, the magnetic energy of the minimizer satisfies*

$$(\kappa H)^2 \int_\Omega |\operatorname{curl} \mathbf{A} - B_0|^2 \, dx = o\left(\kappa H \ln \frac{\kappa}{H}\right), \qquad \text{as } \kappa \longrightarrow +\infty. \tag{3.1.15}$$

If $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$, we introduce the energy density,

$$e(\psi, \mathbf{A}) = |(\nabla - i\kappa H\mathbf{A})\psi|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2.$$

We also introduce the local energy of $(\psi, \mathbf{A})$ in a domain $\overline{\mathcal{D}} \subset \Omega$:

$$\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{D}) = \int_{\mathcal{D}} e(\psi, \mathbf{A}) \, dx. \tag{3.1.16}$$

Furthermore, we define the Ginzburg-Landau energy of $(\psi, \mathbf{A})$ in a domain $\overline{\mathcal{D}} \subset \Omega$ as follows,

$$\mathcal{E}(\psi, \mathbf{A}; \mathcal{D}) = \mathcal{E}_0(u, \mathbf{A}; \mathcal{D}) + (\kappa H)^2 \int_\Omega |\operatorname{curl}(\mathbf{A} - \mathbf{F})|^2 \, dx. \tag{3.1.17}$$

If $\mathcal{D} = \Omega$, we sometimes omit the dependence on the domain and write $\mathcal{E}_0(\psi, \mathbf{A})$ for $\mathcal{E}_0(\psi, \mathbf{A}; \Omega)$.

The next theorem gives a local version of Theorem 3.1.1.

Theorem 3.1.5. *Under Assumption* (3.1.2)*, if* $(\psi, \mathbf{A})$ *is a minimizer of* (3.1.1) *and* $\mathcal{D}$ *is regular set such that* $\overline{\mathcal{D}} \subset \Omega$*, then the following is true.*

1. *If* $H(\kappa)$ *satisfies* (3.1.7)*, then,*

$$\mathcal{E}(\psi, \mathbf{A}, \mathcal{D}) \geq \kappa^2 \int_{\mathcal{D}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx + o\left(\kappa H \ln \frac{\kappa}{H}\right), \quad \text{as } \kappa \longrightarrow +\infty. \tag{3.1.18}$$

2. *If* $H(\kappa)$ *satisfies*

$$C^1_{min}\kappa^{\frac{3}{5}} \leq H \ll \kappa \qquad \text{as } \kappa \longrightarrow +\infty, \tag{3.1.19}$$

where C^1_{min} is a positive constant, then

$$\mathcal{E}(\psi,\mathbf{A},\mathcal{D}) \leq \kappa^2 \int_{\mathcal{D}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx + o\left(\kappa H \ln\frac{\kappa}{H}\right), \quad as\ \kappa \longrightarrow +\infty. \tag{3.1.20}$$

As a consequence of the proof of Theorem 3.1.1, the methods used in [46] allow us to obtain information regarding the distribution of vortices in Ω. When the magnetic field is constant (i.e B_0 is a constant), it is proved in [46] that ψ has vortices whose density tends to be uniform. In Section 3.7 we will prove that, if $(\psi,\mathbf{A})$ is a minimizer of (3.1.1) and $B_0(x)$ is a variable magnetic field, then, ψ has vortices that are distributed everywhere in Ω but with a non uniform density.

The next theorem was proved by E. Sandier and S. Serfaty in [46] when the magnetic field is constant ($B_0(x)=1$).

Theorem 3.1.6. *Suppose that Assumption* (3.1.2) *holds and that $H(\kappa)$ satisfies* (3.1.7)*. Let $(\psi,\mathbf{A})$ be a minimizer of* (3.1.1)*. Then there exists $m=m(\kappa)$ disjoint disks $(D_i(a_i,r_i))_{i=1}^m$ in Ω such that, as $\kappa \longrightarrow +\infty$,*

1. $\sum_{i=1}^m r_i \leq (\kappa H)^{\frac12}\left(\ln\frac{\kappa}{H}\right)^{-\frac74}\int_\Omega \frac{1}{\sqrt{|B_0(x)|}}\,dx\,(1+o(1))$.
2. $|\psi| \geq \frac12$ *on* $\cup_i \partial D_i$.
3. *If* $d_i = \deg\left(\frac{\psi}{|\psi|},\partial D_i\right)$ *is the winding number of* $\frac{\psi}{|\psi|}$ *on* ∂D_i*, then as* $\kappa \longrightarrow +\infty$

$$\mu_\kappa = \frac{2\pi}{\kappa H}\sum_{i=1}^m d_i\delta_{a_i} \longrightarrow B_0(x)\,dx \qquad \text{and} \qquad |\mu_\kappa| = \frac{2\pi}{\kappa H}\sum_{i=1}^m |d_i|\delta_{a_i} \longrightarrow |B_0(x)|\,dx,$$

in the weak sense of measures [2]*, where dx is the Lebesgue measure on $\mathbb{R}^2$ restricted to Ω.*

The measure μ describes the distribution of vortices see Fig.3.1, and it is called the *vorticity measure*, the function $o(1)$ is bounded independently of the choice of the minimizer $(\psi,\mathbf{A})$.

Notation.

Throughout the paper, we use the following notation :

- If $a(\kappa)$ and $b(\kappa)$ are two positive functions, we write $a(\kappa) \ll b(\kappa)$ if $a(\kappa)/b(\kappa) \to 0$ as $\kappa \to \infty$.
- If $a(\kappa)$ and $b(\kappa)$ are two functions with $b(\kappa) \neq 0$, we write $a(\kappa) \sim b(\kappa)$ if $a(\kappa)/b(\kappa) \to 1$ as $\kappa \to \infty$.
- If $a(\kappa)$ and $b(\kappa)$ are two positive functions, we write $a(\kappa) \approx b(\kappa)$ if there exist positive constants c_1, c_2 and κ_0 such that $c_1 b(\kappa) \leq a(\kappa) \leq c_2 b(\kappa)$ for all $\kappa \geq \kappa_0$.

[2] μ_κ converge weakly to μ means that :

$$\mu_\kappa(f) \longrightarrow \mu(f), \qquad \forall f \in C_0(\Omega).$$

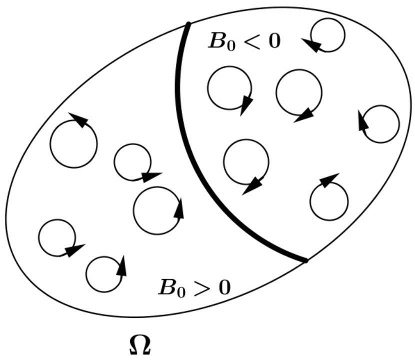

FIGURE 3.1 – Vortices.

- Given $R > 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$, $Q_R(x) = (-R/2+x_1, R/2+x_1)\times(-R/2+x_2, R/2+x_2)$ denotes the square of side length R centered at x and we write $Q_R = Q_R(0)$.

3.2 A reference problem

In this section, we will introduce the function $\hat{f}$ appearing in Theorem 3.1.1, and recall its main properties as proved previously in [45] and [3, 18]. In addition, we will give a new estimate on $\hat{f}$ in Proposition 3.2.4 below.

Consider two constants $b \in (0,1)$ and $R > 0$. If $u \in H^1(Q_R)$, we define the following Ginzburg-Landau energy,

$$F^{\sigma}_{b,Q_R}(u) = \int_{Q_R} \left(b|(\nabla - i\sigma\mathbf{A}_0)u|^2 + \frac{1}{2}\left(1-|u|^2\right)^2 \right) dx\,, \tag{3.2.1}$$

where $\sigma \in \{-1, +1\}$ and

$$\mathbf{A}_0(x) = \frac{1}{2}(-x_2, x_1)\,, \qquad \forall\, x = (x_1, x_2) \in \mathbb{R}^2\,. \tag{3.2.2}$$

Notice that the magnetic potential $\mathbf{A}_0$ satisfies :

$$\operatorname{curl}\mathbf{A}_0 = 1 \quad \text{in } \mathbb{R}^2\,.$$

We introduce the two ground state energies

$$e_N(b,R) = \inf\left\{ F^{+1}_{b,Q_R}(u) : u \in H^1(Q_R;\mathbb{C}) \right\} \tag{3.2.3}$$

$$e_D(b,R) = \inf\left\{ F^{+1}_{b,Q_R}(u) : u \in H^1_0(Q_R;\mathbb{C}) \right\}\,. \tag{3.2.4}$$

The minimization of the functional F^{+1}_{b,Q_R} over '*magnetic periodic*' functions appears naturally in the proof. Let us introduce the following space

$$E_R = \left\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{C}) : u(x_1+R,x_2) = e^{iR\frac{x_2}{2}}u(x_1,x_2), u(x_1,x_2+R) = e^{-iR\frac{x_1}{2}}u(x_1,x_2) \right\}, \tag{3.2.5}$$

together with the ground state energy

$$e_p(b,R) = \inf\left\{ F^{+1}_{b,Q_R}(u) : u \in E_R \right\}. \tag{3.2.6}$$

Since F^{+1}_{b,Q_R} is bounded from below, there exists for each $e_\#(b,R)$ with $\# \in \{N,D,p\}$, a ground state (minimizer). Note also that by comparison of the three domains of minimization it is clear that

$$e_N(b,R) \le e_p(b,R) \le e_D(b,R). \tag{3.2.7}$$

In the three cases, if u is such a ground state, u satisfies the Ginzburg-Landau equation

$$b(\nabla - i\mathbf{A}_0)^2 u = (1-|u|^2)u,$$

and it results from a standard application of the maximum principle that

$$|u| \le 1. \tag{3.2.8}$$

As $F^{+1}_{b,Q_R}(u) = F^{-1}_{b,Q_R}(\overline{u})$, it is also immediate that,

$$\inf_{u\in H^1(Q_R;\mathbb{C})} F^{+1}_{b,Q_R}(u) = \inf_{u\in H^1(Q_R;\mathbb{C})} F^{-1}_{b,Q_R}(u). \tag{3.2.9}$$

In the next theorem we will define the limiting function $\hat{f}$, which describes the ground state energy of both two and three dimensional superconductors subject to high magnetic fields (see [18]).

Theorem 3.2.1. *Let $e_p(b,R)$ be as introduced in* (3.2.6).

1. *For any $b \in [0,\infty)$, there exists a constant $\hat{f}(b) \ge 0$ such that*

$$\hat{f}(b) = \lim_{R\longrightarrow\infty} \frac{e_p(b,R)}{|Q_R|} = \lim_{R\longrightarrow\infty} \frac{e_D(b,R)}{|Q_R|}. \tag{3.2.10}$$

2. *For all $b \ge 1$, $\hat{f}(b) = \frac{1}{2}$.*
3. *The function $[0,\infty) \ni b \longmapsto \hat{f}(b)$ is continuous, non-decreasing and its range is the interval $[0,1/2]$.*
4. *As $b \longrightarrow 0_+$, $\hat{f}(b)$ satisfies*

$$\hat{f}(b) = \frac{b}{2}\ln\frac{1}{b}\,(1+\hat{s}(b)), \tag{3.2.11}$$

where the function $\hat{s} : (0,+\infty) \longmapsto (-\infty,+\infty)$ satisfies

$$\lim_{b\longrightarrow 0} \hat{s}(b) \longrightarrow 0.$$

5. *There exist universal constants C and R_0 such that*

$$\forall R \geq R_0, \qquad \forall b \in [0,1], \qquad \left| \hat{f}(b) - \frac{e_p(b,R)}{R^2} \right| \leq \frac{C}{R}. \tag{3.2.12}$$

6. *There exist positive constants b_0, R_0 and a function*

$$\mathrm{err} : (0,1) \times (0,+\infty) \longrightarrow (0,+\infty)\,, \tag{3.2.13}$$

such that

$$\forall \epsilon \geq 0\,, \exists \eta \geq 0 \text{ if } |b| + \frac{1}{R} < \eta \text{ then } |err(b,R)| < \epsilon\,, \tag{3.2.14}$$

and

$$\forall b \in (0,b_0)\,, \qquad \forall R \in (R_0,+\infty)\,, \qquad \frac{e_N(b,R)}{R^2} \geq \hat{f}(b)(1 - \mathrm{err}(b,R))\,. \tag{3.2.15}$$

The limiting function $\hat{f}$ was defined in ([3], [45], [31]). The estimate in (3.2.11) and (3.2.12) are obtained by Fournais-Kachmar (see [18, Theorem 2.1 and Proposition 2.8]) and by Kachmar (see [31, Theorem 2.4]) respectively. The lower bound in (3.2.15) is a consequence of [31, Theorem 2.1 and (2.9)].

The next proposition gives us a lower bound of the ground state energy $e_N(b,R)$ which is needed it in the proof of Proposition 3.5.1.

Proposition 3.2.2. *There exists a positive constant C, such that if*

$$R \geq 1 \qquad \text{and} \qquad 0 < b < 1\,, \tag{3.2.16}$$

then,

$$e_N(b,R) \leq e_D(b,R) - C R\, b^{\frac{1}{2}}\,. \tag{3.2.17}$$

Proof. Without loss of generality, we can suppose $\sigma = +1$. Let $u \in H^1(Q_R)$ be a minimizer of the functional in (3.2.1), i.e. such that :

$$e_N(b,R) = F^{+1}_{b,Q_R}(u) = \int_{Q_R} \left(b|(\nabla - i\mathbf{A}_0)u|^2 + \frac{1}{2}\left(1-|u|^2\right)^2 \right) dx\,. \tag{3.2.18}$$

We introduce a cut-off function $\chi_{R,b} \in C_c^\infty(\mathbb{R}^2)$ such that

$$0 \leq \chi_{R,b} \leq 1 \qquad \text{in } \mathbb{R}^2\,, \qquad \mathrm{supp}\chi_{R,b} \subset Q_R\,, \qquad \chi_{R,b} = 1 \qquad \text{in } Q_{R-b^{\frac{1}{2}}}\,. \tag{3.2.19}$$

In addition, the function $\chi_{R,b}$ can be chosen such that for some universal constants C and C', we have,

$$|\nabla\chi_{R,b}| \leq C b^{-\frac{1}{2}} \qquad \text{and} \qquad |\Delta\chi_{R,b}| \leq C' b^{-1}\,, \qquad \forall R \geq 1 \qquad \text{and} \qquad \forall b \in (0,1)\,. \tag{3.2.20}$$

Let $u_{R,b}(x)=\chi_{R,b}(x)u(x)$. Then $u_{R,b}\in H^1_0(Q_R)$ and consequently

$$e_D(b,R)\leq F^{+1}_{b,Q_R}(u_{R,b})\,. \tag{3.2.21}$$

We rewrite $F^{+1}_{b,Q_R}(u_{R,b})$ as follows,

$$\begin{aligned}F^{+1}_{b,Q_R}(u_{R,b})&=\int_{Q_R}\left(b|(\nabla-i\mathbf{A}_0)\chi_{R,b}u|^2+\frac12\left(1-|\chi_{R,b}u|^2\right)^2\right)dx\\&=\int_{Q_R}\left(b|(\nabla-i\mathbf{A}_0)\chi_{R,b}u|^2+\frac12\Big(1-2|u|^2+|\chi_{R,b}u|^4+2(|u|^2-|\chi_{R,b}u|^2)\Big)\right)dx\\&\leq\int_{Q_R}\left(b|(\nabla-i\mathbf{A}_0)\chi_{R,b}u|^2+\frac12\Big(1-|u|^2\Big)^2\right)dx+\int_{Q_R\setminus Q_{R-b^{\frac12}}}\Big(1-|\chi_{R,b}|^2\Big)|u|^2\,dx\,.\end{aligned} \tag{3.2.22}$$

We estimate from above the term $\displaystyle\int_{Q_R}|(\nabla-i\mathbf{A}_0)\chi_{R,b}u|^2\,dx$ as follows :

$$\begin{aligned}\int_{Q_R}|(\nabla-i\mathbf{A}_0)\chi_{R,b}u|^2\,dx&=\Big\langle(\nabla-i\mathbf{A}_0)\chi_{R,b}u,(\nabla-i\mathbf{A}_0)\chi_{R,b}u\Big\rangle\\&=\Big\langle\nabla\chi_{R,b}u+\chi_{R,b}(\nabla-i\mathbf{A}_0)u,\nabla\chi_{R,b}u+\chi_{R,b}(\nabla-i\mathbf{A}_0)u\Big\rangle\\&=\Big\langle\nabla\chi_{R,b}u,\nabla\chi_{R,b}u\Big\rangle+\Big\langle\chi_{R,b}(\nabla-i\mathbf{A}_0)u,\chi_{R,b}(\nabla-i\mathbf{A}_0)u\Big\rangle\\&\quad+\Big\langle\nabla\chi_{R,b}u,\chi_{R,b}(\nabla-i\mathbf{A}_0)u\Big\rangle+\Big\langle\chi_{R,b}(\nabla-i\mathbf{A}_0)u,\nabla\chi_{R,b}u\Big\rangle\,.\end{aligned}$$

An integration by parts yields,

$$\begin{aligned}\langle\nabla\chi_{R,b}u,\chi_{R,b}(\nabla-i\mathbf{A}_0)u\rangle&=-\langle\nabla\chi_{R,b}u,\nabla\chi_{R,b}u\rangle\\&\quad-\langle\chi_{R,b}\Delta\chi_{R,b}u,u\rangle-\langle\chi_{R,b}(\nabla-i\mathbf{A}_0)u,\nabla\chi_{R,b}u\rangle\,,\end{aligned} \tag{3.2.23}$$

which implies that

$$\int_{Q_R}|(\nabla-i\mathbf{A}_0)\chi_{R,b}u|^2\,dx=\Big\langle\chi_{R,b}(\nabla-i\mathbf{A}_0)u,\chi_{R,b}(\nabla-i\mathbf{A}_0)u\Big\rangle-\Big\langle\chi_{R,b}\Delta\chi_{R,b}u,u\Big\rangle. \tag{3.2.24}$$

Putting (3.2.24) into (3.2.22), we get

$$\begin{aligned}F^{+1}_{b,Q_R}(u_{R,b})&\leq\int_{Q_R}\left(b|\chi_{R,b}(\nabla-i\mathbf{A}_0)u|^2+\frac12\Big(1-|u|^2\Big)^2\right)dx\\&\quad+\int_{Q_R\setminus Q_{R-b^{\frac12}}}\Big(1-|\chi_{R,b}|^2\Big)|u|^2\,dx+b\int_{Q_R\setminus Q_{R-b^{\frac12}}}|\Delta\chi_{R,b}|\,|u|^2\,dx\,.\end{aligned} \tag{3.2.25}$$

By using the bound $|u|\leq1$, (3.2.20) and the assumption on the support of $\chi_{R,b}$ in (3.2.19), it is easy to check that,

$$F^{+1}_{b,Q_R}(u_{R,b}) \leq F^{+1}_{b,Q_R}(u) + C R\, b^{\frac{1}{2}}\,.$$

Using (3.2.21) and (3.2.18), we get

$$e_D(b,R) \leq e_N(b,R) + C R\, b^{\frac{1}{2}}\,.$$

□

Corollary 3.2.3. *With $\hat{f}(b)$ introduced in* (3.2.10)*, it holds,*

$$\hat{f}(b) = \lim_{R\longrightarrow+\infty} \frac{e_N(b,R)}{R^2}\,. \tag{3.2.26}$$

Proof. We have from (3.2.7) and (3.2.17) that, for any $b \in (0,1)$,

$$e_D(b,R) - C R\, b^{\frac{1}{2}} \leq e_N(b,R) \leq e_D(b,R)\,.$$

Having in mind (3.2.10), we divide all sides of this inequality by R^2 and then take the limit as $R \longrightarrow +\infty$. That gives us

$$\hat{f}(b) = \lim_{R\longrightarrow\infty} \frac{e_N(b,R)}{R^2}\,.$$

□

Proposition 3.2.4 (Fournais)**.** *There exists a positive constant C, such that if* (3.2.16) *is satisfied, then*

$$\frac{e_D(b,R)}{R^2} \leq \hat{f}(b) + C\frac{\sqrt{b}}{R}\,, \tag{3.2.27}$$

$$\frac{e_D(b,R)}{R^2} \geq \hat{f}(b)\,. \tag{3.2.28}$$

Proof. We have already seen that

$$\hat{f}(b) = \lim_{R\to+\infty} \frac{e_D(b,R)}{R^2}\,. \tag{3.2.29}$$

Let us first prove (3.2.28). Let $n \in \mathbb{N}^*$ and $R > 0$. Let $u \in H^1_0(Q_R)$ be a minimizer of F^{+1}_{b,Q_R} (i.e. $e_D(b,R) = F^{+1}_{b,Q_R}(u)$). We extend u to a function $\widetilde{u} \in H^1_0(Q_{nR})$ by '*magnetic periodicity*' as follows

$$\widetilde{u}(x_1+R,x_2) = e^{iR\frac{x_2}{2}}u(x_1,x_2)\,, \qquad \widetilde{u}(x_1,x_2+R) = e^{-iR\frac{x_1}{2}}u(x_1,x_2)\,.$$

Let $\mathcal{J}^n = \{j \in \mathbb{Z}, 1 \leq j \leq n^2\}$. Notice that, the square Q_{nR} is formed exactly of n^2 squares $(Q_R(x_0^j))_{j\in\mathcal{J}^n}$. We define in each $Q_R(x_0^j)$ the following function

$$u_j = \widetilde{u}_{|_{Q_R(x_0^j)}}\,.$$

Observe that u_j is a minimizer of $F^{+1}_{b,Q_R(x_0^j)}$ in $H^1_0(Q_R(x_0^j))$ and if we extend u_j by 0 outside of

$Q_R(x_0^j)$, keeping the same notation u_j for this extension, we have, $\widetilde{u} = \sum_{i\in\mathcal{J}^n} u_j$. Using magnetic translation invariance, it is easy to check that

$$F^{+1}_{b,Q_{nR}}(\widetilde{u}) = \sum_{j\in\mathcal{J}^n} F^{+1}_{b,Q_R}(u_j) = n^2 e_D(b,R)\,.$$

Consequently, we get

$$e_D(b,nR) \leq n^2 e_D(b,R)\,.$$

We now divide both sides of this inequality by n^2R^2 then we take the limit as $n \longrightarrow \infty$. Having in mind (3.2.10), this gives (3.2.28).

We prove (3.2.27).

If $n \in \mathbb{N}^*$ and $j = (j_1, j_2) \in \mathbb{Z}^2$, we denote by

$$K_j = I_{j_1} \times I_{j_2}\,,$$

where

$$\forall m \in \mathbb{Z}, \qquad I_m = \left(\frac{2m+1-n}{2} - \frac{1}{2}, \frac{2m+1-n}{2} + \frac{1}{2}\right)\,.$$

For all $R > 0$, we set

$$Q_{R,j} = \{Rx : \, x \in K_j\}\,.$$

Let $\mathcal{J}^n = \{j = (j_1,j_2) \in \mathbb{Z}^2 : 0 \leq j_1, j_2 \leq n-1\}$ and $Q_{nR} = \left(-\frac{nR}{2}, \frac{nR}{2}\right) \times \left(-\frac{nR}{2}, \frac{nR}{2}\right)$. Then the family $(\overline{Q}_{R,j})$ is a covering of Q_{nR}, formed exactly of n^2 squares. Let $u = u_{nR} \in H^1_0(Q_{nR})$ be a minimizer of of $F^{+1}_{b,Q_{nR}}$ i.e. $F^{+1}_{b,Q_{nR}}(u) = e_D(b,nR)$. We have the obvious decomposition,

$$\int_{Q_{nR}} |u(x)|^4\,dx = \sum_{i\in\mathcal{J}^n} \int_{Q_{R,j}} |u(x)|^4\,dx\,. \tag{3.2.30}$$

Let $\chi = \chi_{R,b^{\frac12}}(x - x_0^j)$, where $\chi_{R,b^{\frac12}}$ is the cut-off function introduced in (3.2.19). The function u satisfies $-b(\nabla - i\mathbf{A}_0)^2 u = (1-|u|^2)u$ in Q_{nR}. It results from an integration by parts that

$$e_D(b,nR) = F^{+1}_{b,Q_{nR}}(u) = -\frac{1}{2}\int_{Q_{nR}} (|u(x)|^4 - 1)\,dx\,. \tag{3.2.31}$$

We may write,

$$\begin{aligned}
\int_{Q_{R,j}} |(\nabla - i\mathbf{A}_0)\chi u|^2\,dx &= \Big\langle (\nabla - i\mathbf{A}_0)\chi u, (\nabla - i\mathbf{A}_0)\chi u\Big\rangle \\
&= \Big\langle \nabla\chi\, u, \nabla\chi\, u\Big\rangle + \Big\langle \chi\,(\nabla - i\mathbf{A}_0)u, \chi\,(\nabla - i\mathbf{A}_0)u\Big\rangle \\
&\qquad + 2\Big\langle \nabla\chi\, u, \chi\,(\nabla - i\mathbf{A}_0)u\Big\rangle \\
&= \Big\langle \nabla\chi\, u, \nabla\chi\, u\Big\rangle + \Big\langle (\nabla - i\mathbf{A}_0)(\chi^2\, u), (\nabla - i\mathbf{A}_0)u\Big\rangle.
\end{aligned}$$

An integration by parts gives us

$$\int_{Q_{R,j}} |(\nabla - i\mathbf{A}_0)\chi u|^2\,dx = \int_{Q_{R,j}} |\nabla\chi|^2|u|^2\,dx - \Big\langle \chi^2\,u, (\nabla - i\mathbf{A}_0)^2 u\Big\rangle. \tag{3.2.32}$$

Using (3.2.32), we may express the energy $F^{+1}_{b,Q_{R,j}}(\chi u)$ as follows :

$$\begin{aligned}
F^{+1}_{b,Q_{R,j}}(\chi u) &= \int_{Q_{R,j}} \Big(b|(\nabla - i\mathbf{A}_0)\chi u|^2 - |\chi u|^2\Big)\,dx + \frac{1}{2}\int_{Q_{R,j}} \Big(|\chi u|^4 + 1\Big)dx \\
&= -\langle \chi^2 u, (b(\nabla - i\mathbf{A}_0)^2 + 1)u\rangle + b\int_{Q_{R,j}} |\nabla\chi|^2|u|^2\,dx + \frac{1}{2}\int_{Q_{R,j}} (\chi^4|u|^4 + 1)dx\,.
\end{aligned}$$

Using the equation $(b(\nabla - i\mathbf{A}_0)^2 + 1)u = |u|^2 u$ and the inequality $\chi^4 \le \chi^2$, we get

$$\begin{aligned}
F^{+1}_{b,Q_{R,j}}(\chi u) &\le b\int_{Q_{R,j}} |\nabla\chi|^2|u|^2\,dx - \frac{1}{2}\int_{Q_{R,j}} \Big(\chi^2|u|^4 - 1\Big)dx \\
&\le b\int_{Q_{R,j}} |\nabla\chi|^2|u|^2\,dx + \frac{1}{2}\int_{Q_{R,j}} \Big(1 - \chi^2\Big)\,dx - \frac{1}{2}\int_{Q_{R,j}} \Big(|u|^4 - 1\Big)dx \\
&\le -\frac{1}{2}\int_{Q_{R,j}} \Big(|u|^4 - 1\Big)\,dx + Cb^{\frac{1}{2}}R\,.
\end{aligned}$$

Since each χu has support in a square of side length R, we get

$$F^{+1}_{b,Q_{R,j}}(\chi u) \ge e_D(b,R)\,. \tag{3.2.33}$$

We sum over the n^2 squares $(Q_{R,j})_{j\in\mathcal{J}^n}$ (that cover Q_{nR}), and get

$$n^2 e_D(b,R) \le -\frac{1}{2}\int_{Q_{nR}} (|u|^4 - 1)\,dx + Cb^{\frac{1}{2}}Rn^2\,.$$

Using (3.2.31), we obtain

$$n^2 e_D(b,R) \le e_D(b,nR) + Cn^2Rb^{\frac{1}{2}}\,.$$

Dividing by n^2R^2, we obtain

$$\frac{e_D(b,R)}{R^2} \le \frac{e_D(b,nR)}{(nR)^2} + CR^{-1}b^{\frac{1}{2}}\,.$$

We take the limit $n \to +\infty$ and get (3.2.27). □

3.3 Upper bound of the energy

The aim of this section is to give an upper bound on the ground state energy $\mathrm{E_g}(\kappa, H)$ introduced in (3.1.6).

In the sequel, for some choice of $\rho \in (0,1)$ to be determined later (see (3.3.11)), we consider triples $(\ell, x_0, \widetilde{x}_0)$ such that $\overline{Q_\ell(x_0)} \subset \{|B_0| > \rho\} \cap \Omega$ and $\widetilde{x}_0 \in \overline{Q_\ell(x_0)}$. In this situation, we say that this triple is ρ-admissible, that the pair (ℓ, x_0) is ρ-admissible and the corresponding square

$Q_\ell(x_0)$ is a ρ-admissiblle. Let

$$R=\ell\sqrt{\kappa H|B_0(\widetilde{x}_0)|} \tag{3.3.1}$$

We introduce the function :

$$w_{\ell,x_0,\widetilde{x}_0}(x)=\begin{cases} e^{i\kappa H\varphi_{x_0,\widetilde{x}_0}}u_R\left(\frac{R}{\ell}(x-x_0)\right) & \text{if } x\in Q_\ell(x_0)\subset\{B_0>\rho\}\cap\Omega\\ e^{i\kappa H\varphi_{x_0,\widetilde{x}_0}}\overline{u}_R\left(\frac{R}{\ell}(x-x_0)\right) & \text{if } x\in Q_\ell(x_0)\subset\{B_0<-\rho\}\cap\Omega\,,\end{cases} \tag{3.3.2}$$

where $u_R\in H^1_0(\Omega)$ is a minimizer of the functional in (3.2.1) and $\varphi_{x_0,\widetilde{x}_0}$ is the function introduced in [5, Lemma A.3] that satisfies

$$|\mathbf{F}(x)-\sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0)-\nabla\varphi_{x_0,\widetilde{x}_0}(x)|\leq C\ell^2,\qquad (x\in Q_\ell(x_0))\ , \tag{3.3.3}$$

where $B_0(\widetilde{x}_0)=\operatorname{curl}\mathbf{F}(\widetilde{x}_0)$, $\mathbf{A}_0$ is the magnetic potential introduced in (3.2.2) and σ_ℓ is the sign of $B_0(x)$ in $Q_\ell(x_0)$.

Proposition 3.3.1. *Under Assumption* (3.1.2)*, there exist positive constants C and κ_0 such that if $\kappa\geq\kappa_0$, $\ell\in(0,1)$, $\delta\in(0,1)$, $\rho>0$, $\ell^2\kappa H\rho>1$, and $(\ell,x_0,\widetilde{x}_0)$ is a ρ-admissible triple, then,*

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0},\mathbf{F},Q_\ell(x_0))\leq(1+\delta)\kappa^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)+C\left(\frac{1}{\ell H}+\delta^{-1}\ell^4\kappa H\right)\kappa H\,. \tag{3.3.4}$$

Proof. Let

$$b=\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\,. \tag{3.3.5}$$

We estimate $\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0},\mathbf{F},Q_\ell(x_0))$ from above. We write for any $\delta\in(0,1)$

$$\begin{aligned}
&\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0},\mathbf{F},Q_\ell(x_0))\\
&\qquad=\int_{Q_\ell(x_0)}\left[|(\nabla-i\kappa H\mathbf{F})w_{\ell,x_0,\widetilde{x}_0}|^2+\frac{\kappa^2}{2}(1-|w_{\ell,x_0,\widetilde{x}_0}|^2)^2\right]dx\\
&\qquad\leq(1+\delta)\int_{Q_\ell(x_0)}\Big[|(\nabla-i\kappa H(\sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0)+\nabla\varphi_{x_0,\widetilde{x}_0}(x)))w_{\ell,x_0,\widetilde{x}_0}|^2\\
&\qquad\qquad+\frac{\kappa^2}{2}(1-|w_{\ell,x_0,\widetilde{x}_0}|^2)^2\Big]dx\\
&\qquad\quad+C(\kappa H)^2\delta^{-1}\int_{Q_\ell(x_0)}\left|\mathbf{F}-(\sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0)+\nabla\varphi_{x_0,\widetilde{x}_0}(x))w_{\ell,x_0,\widetilde{x}_0}\right|^2dx\\
&\qquad\leq(1+\delta)\mathcal{E}_0\big(w_{\ell,x_0,\widetilde{x}_0},\,\sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0)+\nabla\varphi_{x_0,\widetilde{x}_0}(x),\,Q_\ell(x_0)\big)+C\delta^{-1}\ell^6(\kappa H)^2.
\end{aligned} \tag{3.3.6}$$

Using (3.2.9), the definition of $w_{\ell,x_0,\widetilde{x}_0}$ and the change of variable $y = \frac{R}{\ell}(x - x_0)$, we obtain

$$\begin{aligned}
&\mathcal{E}_0\big(w_{\ell,x_0,\widetilde{x}_0},\, \sigma_\ell |B_0(\widetilde{x}_0)| \mathbf{A}_0(x - x_0) + \nabla\varphi_{x_0,\widetilde{x}_0}(x),\, Q_\ell(x_0)\big) \\
&= \int_{Q_R} \left[\left| \left(\frac{R}{\ell}\nabla_y - i\frac{R}{\ell}\mathbf{A}_0(y) \right) u_R(y) \right|^2 + \frac{\kappa^2}{2}\left(1 - |u_R(y)|^2\right)^2 \right] \frac{\ell^2}{R^2}\, dy \\
&= \frac{1}{b} F^{+1}_{b,Q_R}(u_R)\,. \qquad (3.3.7)
\end{aligned}$$

Since $u_R \in H^1_0(Q_R)$ is a minimizer of F^{+1}_{b,Q_R}, then

$$F^{+1}_{b,Q_R}(u_R) = e_D(b,R)\,. \qquad (3.3.8)$$

Proposition 3.2.4 tells us that $\dfrac{e_D(b,R)}{R^2} \leq \hat{f}(b) + C\dfrac{\sqrt{b}}{R}$ for all $b \in]0,1[$ and $R \geq 1$. This assumption is satisfied because $R \geq \ell\sqrt{\kappa H \rho} > 1$ (see Remark 3.3.2). Therefore, we get from (3.3.7) and (3.3.8) the estimate

$$\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0}, \sigma_\ell |B_0(\widetilde{x}_0)| \mathbf{A}_0(x - x_0) + \nabla\varphi_{x_0,\widetilde{x}_0}(x), Q_\ell(x_0)) \leq R^2 \frac{\hat{f}(b)}{b} + C\frac{R}{\sqrt{b}}\,, \qquad (3.3.9)$$

with b defined in (3.3.5).
We get by collecting the estimates in (3.3.6)-(3.3.7) that,

$$\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0}, \mathbf{F}, Q_\ell(x_0)) \leq (1+\delta)R^2\frac{\hat{f}(b)}{b} + C_1\frac{R}{\sqrt{b}} + C_2\delta^{-1}\ell^6(\kappa H)^2\,. \qquad (3.3.10)$$

Remembering the definition of R and b in (3.3.1) and (3.3.5) respectively, we get

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0}, \mathbf{F}, Q_\ell(x_0)) \leq (1+\delta)\kappa^2 \hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right) + C\left(\frac{\kappa}{\ell} + \delta^{-1}\ell^4(\kappa H)^2\right)\,,$$

which finishes the proof of Proposition 3.3.1. □

Remark 3.3.2. We select ℓ, δ and ρ as follow :

$$\ell = (\kappa H)^{-\frac{1}{4}}\,, \qquad \rho = (\kappa H)^{-\frac{1}{3}}\,. \qquad (3.3.11)$$

and

$$\delta = \left(\ln\frac{\kappa}{H}\right)^{-\frac{1}{4}} \qquad (3.3.12)$$

Under Assumption (3.1.7), this choice permits us to verify the assumptions in Proposition 3.3.1 and to obtain error terms of order $o\left(\kappa H \ln\frac{\kappa}{H}\right)$. We have indeed as $\kappa \longrightarrow +\infty$

$$\frac{\kappa}{\ell\kappa H \ln\frac{\kappa}{H}} = \frac{\kappa^{\frac{1}{4}}}{H^{\frac{3}{4}}\ln\frac{\kappa}{H}} \ll 1\,,$$

$$\frac{\delta^{-1}(\kappa H)^2\ell^4}{\kappa H\ln\frac{\kappa}{H}} = \frac{1}{\left(\ln\frac{\kappa}{H}\right)^{\frac{3}{4}}} \ll 1\,,$$

$$\ell(\kappa H)^{\frac{1}{2}}\rho^{\frac{1}{2}} = (\kappa H)^{\frac{1}{12}} \gg 1\,.$$

Theorem 3.3.3. *Under Assumption* (3.1.2), *if* (3.1.7) *holds, then, the ground state energy* $\mathrm{E_g}(\kappa,H)$ *in* (3.1.6) *satisfies*

$$\mathrm{E_g}(\kappa,H) \le \kappa^2\int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)dx + o\left(\kappa H\ln\frac{\kappa}{H}\right)\,,\quad \textit{as } \kappa\longrightarrow+\infty\,. \tag{3.3.13}$$

Proof. Let $\ell\in(0,1)$, δ and ρ be the parameters depending on κ and chosen as in Remark 3.3.2. As we did in the previous paper [5, Proposition 5.1], we consider the lattice $\Gamma_\ell := \ell\mathbb{Z}\times\ell\mathbb{Z}$ and write, for $\gamma,\widetilde{\gamma}\in\Gamma_\ell$,

$$Q_{\gamma,\ell} = Q_\ell(\gamma) \text{ and } w_{\ell,x_0,\widetilde{x}_0} = w_{\ell,\gamma,\widetilde{\gamma}}\,.$$

For any $\gamma\in\Gamma_\ell$ such that $Q_{\gamma,\ell}$ is ρ-admissible square, let

$$\underline{B}_{\gamma,\ell} = \inf_{x\in Q_{\gamma,\ell}}|B_0(x)| \tag{3.3.14}$$

and

$$\mathcal{I}_{\ell,\rho} = \left\{\gamma;\,\overline{Q_{\gamma,\ell}}\subset\Omega\cap\{|B_0|>\rho\}\right\}\,,\qquad N = \mathrm{card}\,\mathcal{I}_{\ell,\rho}\,. \tag{3.3.15}$$

Then as $\kappa\to+\infty$, we have :

$$N = |\Omega|\ell^{-2} + \mathcal{O}(\ell^{-1}) + \mathcal{O}(\rho\ell^{-2}). \tag{3.3.16}$$

Note that (3.3.16) results from both conditions in Assumption 3.1.2. For all $x\in\Omega$, we define,

$$s(x) = \sum_{\gamma\in\mathcal{J}_{\ell,\rho}} w_{\ell,\gamma,\widetilde{\gamma}}(x)\,, \tag{3.3.17}$$

where $w_{\ell,\gamma,\widetilde{\gamma}}$ has been extended by 0 outside of $Q_{\gamma,\ell}$. Remember the functional $\mathcal{E}_{\kappa,H}$ in (3.1.1). We compute the energy of the test configuration $(s,\mathbf{F})$. Since $\mathrm{curl}\,\mathbf{F}=B$, we get,

$$\mathcal{E}_{\kappa,H}(s,\mathbf{F},\Omega) = \sum_{\gamma\in\mathcal{J}_{\ell,\rho}}\mathcal{E}_0(w_{\ell,\gamma,\widetilde{\gamma}},\mathbf{F},Q_{\gamma,\ell})\,. \tag{3.3.18}$$

Recall that for any $\widetilde{\gamma}\in Q_{\gamma,\ell}$, $B_0(\widetilde{\gamma})$ satisfies (3.3.3). Then, we select $\widetilde{\gamma}\in Q_{\gamma,\ell}$ such that

$$|B_0(\widetilde{\gamma})| = \underline{B}_{\gamma,\ell}\,.$$

Using Proposition 3.3.1 and noticing that $|Q_{\gamma,\ell}|=\ell^2$, we get for any $\delta\in(0,1)$

$$\sum_{\gamma\in\mathcal{J}_{\ell,\rho}}\mathcal{E}_0(w_{\ell,\gamma,\widetilde{\gamma}},\mathbf{F},Q_{\gamma,\ell}) \le \kappa^2(1+\delta)\sum_{\gamma\in\mathcal{J}_{\ell,\rho}}\hat{f}\left(\frac{H}{\kappa}\underline{B}_{\gamma,\ell}\right)\ell^2 + r(\kappa,H,\ell)\,, \tag{3.3.19}$$

where

$$r(\kappa,H,\ell)=\mathcal{O}\left(\frac{\kappa}{\ell}+\delta^{-1}\ell^4(\kappa H)^2\right)\,. \tag{3.3.20}$$

Having in mind Property (3) of the function $\hat{f}$ established in Theorem 3.2.1, we recognize the lower Riemann sum and notice that $\cup_{\gamma\in\mathcal{J}_{\ell,\rho}}Q_{\gamma,\ell}\subset\Omega$, then, we get by collecting (3.3.18)-(3.3.19) that

$$\mathcal{E}_{\kappa,H}(s,\mathbf{F},\Omega)\leq(1+\delta)\kappa^2\int_\Omega\hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx+r(\kappa,H,\ell)\,. \tag{3.3.21}$$

The choice of the parameters δ in (3.3.12) and ℓ in (3.3.11) implies that all error terms are of lower order compared to $\kappa H\ln\frac{\kappa}{H}$.

□

Remark 3.3.4. The remainder term in (3.3.20) is small compared with the leading order term. We have, for any $\rho_0>0$

$$\begin{aligned}\kappa^2\int_\Omega\hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx&\geq\kappa^2\int_{\Omega\cap\{|B_0|>\rho_0\}}\hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx\\&\geq\kappa^2\hat{f}\left(\frac{H}{\kappa}\rho_0\right)|\Omega\cap\{|B_0|>\rho_0\}|\,.\end{aligned}$$

In view of (3.2.11), for all positive constant C there exists $\rho_0>0$ such that if $H\leq C\kappa$ and $\frac{\rho_0}{C_1}$ is sufficiently small for some $C_1>0$, then as $\kappa\longrightarrow+\infty$

$$\kappa^2\int_\Omega\hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx\geq C_2\frac{\kappa H\rho_0}{2}\ln\frac{\kappa}{H\rho_0}(1+o(1))\,,$$

where C_2 is a positive constant.
In particular, when (3.1.7) is satisfied, we see that,

$$r(\kappa,H,\ell)\ll\kappa^2\int_\Omega\hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx\,. \tag{3.3.22}$$

3.4 A priori estimates of minimizers

The aim of this section is to give a priori estimates on the solutions of the Ginzburg-Landau equations (3.1.5). These estimates play an essential role in controlling the error resulting from various approximations. The starting point is the following L^∞-bound resulting from the maximum principle. If $(\psi,\mathbf{A})\in H^1(\Omega;\mathbb{C})\times H^1_{\mathrm{div}}(\mathbb{R}^2)$ is a solution of (3.1.5), then

$$\|\psi\|_{L^\infty(\Omega)}\leq 1\,. \tag{3.4.1}$$

Next we prove an estimate on the induced magnetic potential.

Proposition 3.4.1. *Suppose that the magnetic field H is a function of κ and satisfies* (3.1.7). *Let $\alpha\in(0,1)$. There exist positive constants κ_0 and C such that, if $\kappa\geq\kappa_0$ and $(\psi,\mathbf{A})$ is a* *minimizer of* (3.1.1)*, then*

$$\|\mathbf{A}-\mathbf{F}\|_{H^2(\Omega)} \le \frac{C}{H}\left(\int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)dx\right)^{\frac{1}{2}},$$

$$\|\mathbf{A}-\mathbf{F}\|_{C^{0,\alpha}(\overline{\Omega})} \le \frac{C}{H}\left(\int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)dx\right)^{\frac{1}{2}}.$$

Here $\mathbf{F}$ *is the magnetic potential introduced in* (3.1.3)*.*

Proof. The estimate in $C^{0,\alpha}$-norm is a consequence of the continuous Sobolev embedding of $H^2(\Omega)$ in $C^{0,\alpha}(\overline{\Omega})$.
It is easy to show that

$$\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)} \le \frac{1}{\kappa H}\,\mathrm{E_g}(\kappa,H)^{\frac{1}{2}}, \tag{3.4.2}$$

and

$$\|(\nabla - i\kappa H\mathbf{A})\psi\|_{L^2(\Omega)} \le \mathrm{E_g}(\kappa,H)^{\frac{1}{2}}. \tag{3.4.3}$$

Notice that under Assumption (3.1.7), it follows from Theorem 3.3.3 and Remark 3.3.4 that

$$\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)} \le \frac{1}{H}\left(\int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)dx\right)^{\frac{1}{2}}, \tag{3.4.4}$$

$$\|(\nabla - i\kappa H\mathbf{A})\psi\|_{L^2(\Omega)} \le \kappa\left(\int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)dx\right)^{\frac{1}{2}}. \tag{3.4.5}$$

Let $a=\mathbf{A}-\mathbf{F}$. We will prove that

$$\|a\|_{H^2(\Omega)} \le \frac{C}{H}\left(\int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)dx\right)^{\frac{1}{2}}.$$

Since $\operatorname{div} a = 0$ and $a\cdot\nu = 0$ on $\partial\Omega$, we get by regularity of the curl-div system see [14, Appendix A.5]

$$\|a\|_{H^2(\Omega)} \le C'\|\operatorname{curl} a\|_{H^1(\Omega)}. \tag{3.4.6}$$

The second equation in (3.1.5) reads as follows :

$$-\nabla^\perp \operatorname{curl} a = \frac{1}{\kappa H}\operatorname{Im}(\overline{\psi}(\nabla - i\kappa H\mathbf{A})\psi).$$

The estimates in (3.4.1) and the bound in (3.4.6), give us

$$\|a\|_{H^2(\Omega)} \le C\left(\|\operatorname{curl} a\|_{L^2(\Omega)} + \frac{1}{\kappa H}\|(\nabla - i\kappa H\mathbf{A})\psi\|_{L^2(\Omega)}\right).$$

Inserting the estimates in (3.4.4) and (3.4.5) into this upper bound finishes the proof of the proposition. □

3.5 Proof of Theorem 3.1.5 : Lower bound

In this section, we suppose that $\mathcal{D}$ is an open set with smooth boundary such that $\overline{\mathcal{D}} \subset \Omega$. We will give a lower bound of the energy $\mathcal{E}(\psi, \mathbf{A}; \mathcal{D})$ introduced in (3.1.17), when $(\psi, \mathbf{A})$ is a minimizer of the functional in (3.1.1).

Construction of a gauge transformation :

Let $\phi_{x_0}(x) = (\mathbf{A}(x_0) - \mathbf{F}(x_0)) \cdot x$, where $\mathbf{F}$ is the magnetic potential introduced in (3.1.3) and (ℓ, x_0) a ρ-admissible pair. Choosing $\alpha \in (0,1)$ and using the estimate of $\|\mathbf{A} - \mathbf{F}\|_{C^{0,\alpha}(\Omega)}$ given in Proposition 3.4.1, we get for all $x \in Q_\ell(x_0)$,

$$\begin{aligned} |\mathbf{A}(x) - \nabla\phi_{x_0} - \mathbf{F}(x)| &= |(\mathbf{A} - \mathbf{F})(x) - (\mathbf{A} - \mathbf{F})(x_0)| \\ &\leq \|\mathbf{A} - \mathbf{F}\|_{C^{0,\alpha}(\Omega)} |x - x_0|^\alpha \\ &\leq C_\alpha \, \lambda \, \ell^\alpha \,, \end{aligned} \tag{3.5.1}$$

where

$$\lambda = \frac{1}{H} \left(\int_\Omega \hat{f} \left(\frac{H}{\kappa} |B_0(x)| \right) dx \right)^{\frac{1}{2}} . \tag{3.5.2}$$

Using (3.2.11), it is clear that, under condition (3.1.7)

$$\lambda^2 = \mathcal{O}\left(\frac{1}{\kappa H} \ln \frac{\kappa}{H} \right) , \tag{3.5.3}$$

as $\kappa \longrightarrow +\infty$.

Proposition 3.5.1. *For all $\alpha \in (0,1)$, there exist positive constants C and κ_0 such that if $\kappa \geq \kappa_0$, $\ell \in (0, \frac{1}{2})$, $\delta \in (0,1)$, $\rho > 0$, $\ell^2 \kappa H \rho > 1$, $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ is a minimizer of (3.1.1), and $(\ell, x_0, \widetilde{x}_0)$ a ρ-admissible triple, then,*

$$\frac{1}{|Q_\ell(x_0)|} \mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0)) \geq (1-\delta)\kappa^2 \hat{f} \left(\frac{H}{\kappa} |B_0(\widetilde{x}_0)| \right) - C \left(\frac{\kappa}{\ell} + \delta^{-1} (\kappa H)^2 \ell^4 + \delta^{-1} \kappa^2 H^2 \lambda^2 \ell^{2\alpha} \right) .$$

Proof. Let $\widetilde{x}_0 \in \overline{Q_\ell(x_0)}$. Recall the function $\varphi_{x_0, \widetilde{x}_0}$ satisfiying (3.3.3). For all $x \in Q_\ell(x_0)$, let

$$u(x) = e^{-i\kappa H \varphi} \psi(x) \,, \tag{3.5.4}$$

where $\varphi = \varphi_{x_0, \widetilde{x}_0} + \phi_{x_0}$ and ϕ_{x_0} is introduced in (3.5.1).

Estimate of $\mathcal{E}_0$ in $Q_\ell(x_0)$:

As we did in [5, Lemma 4.1], we have, for any $\delta \in (0,1)$ and $\alpha \in (0,1)$

$$\begin{aligned} \mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0)) &\geq (1-\delta) \mathcal{E}_0(u, \sigma_\ell |B_0(\widetilde{x}_0)| \mathbf{A}_0(x - x_0); Q_\ell(x_0)) \\ &\quad - C\delta^{-1} (\kappa H)^2 \left(\ell^4 + \lambda^2 \ell^{2\alpha} \right) \int_{Q_\ell(x_0)} |\psi|^2 \, dx. \end{aligned} \tag{3.5.5}$$

Let

$$R = \ell\sqrt{\kappa H|B_0(\widetilde{x}_0)|} \qquad \text{and} \qquad b = \frac{H}{\kappa}|B_0(\widetilde{x}_0)|\,. \tag{3.5.6}$$

Define the function in Q_R

$$v_{\ell,x_0,\widetilde{x}_0}(x) = \begin{cases} u\left(\frac{\ell}{R}x + x_0\right) & \text{if } x \in Q_R \subset \{\{B_0 > \rho\} \cap \Omega\} \\ \overline{u}\left(\frac{\ell}{R}x + x_0\right) & \text{if } x \in Q_R \subset \{\{B_0 < -\rho\} \cap \Omega\}\,. \end{cases} \tag{3.5.7}$$

Using (3.2.9), and the change of variable $y = \frac{R}{\ell}(x - x_0)$, we get

$$\mathcal{E}_0(u, \sigma_\ell\,|B_0(\widetilde{x}_0)|\mathbf{A}_0(x - x_0); Q_\ell(x_0)) = \frac{1}{b}F^{+1}_{b,Q_R}(v_{\ell,x_0,\widetilde{x}_0})\,. \tag{3.5.8}$$

Here F^{+1}_{b,Q_R} is introduced in (3.2.1). Since $v_{\ell,x_0,\widetilde{x}_0} \in H^1(Q_R)$, we have

$$F^{+1}_{b,Q_R}(v_{\ell,x_0,\widetilde{x}_0}) \geq e_N(b, R)\,. \tag{3.5.9}$$

By collecting (3.2.17)-(3.2.28) and the lower bound in (3.5.9), we get,

$$F^{+1}_{b,Q_R}(v_{\ell,x_0,\widetilde{x}_0}) \geq R^2\hat{f}(b) - CRb^{\frac{1}{2}}\,. \tag{3.5.10}$$

As a consequence, (3.5.8) gives us

$$\mathcal{E}_0(u, \sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x - x_0); Q_\ell(x_0)) \geq R^2\frac{\hat{f}(b)}{b} - C\frac{R}{\sqrt{b}}\,. \tag{3.5.11}$$

with b and R introduced in (3.5.6).
Inserting (3.5.11) into (3.5.5) and using the bound of ψ in (3.4.1), we get

$$\mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0)) \geq (1 - \delta)R^2\frac{\hat{f}(b)}{b} - C\frac{R}{\sqrt{b}} - C'\delta^{-1}(\kappa H)^2\left(\ell^4 + \lambda^2\ell^{2\alpha}\right)\ell^2\,.$$

Having in mind (3.5.6), we get for any $\alpha \in (0, 1)$

$$\mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0)) \geq \left((1 - \delta)\kappa^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right) - C\left(\frac{\kappa}{\ell} + \delta^{-1}(\kappa H)^2\ell^4 + \delta^{-1}\kappa^2H^2\lambda^2\ell^{2\alpha}\right)\right)\ell^2\,. \tag{3.5.12}$$

This finishes the proof of Proposition 3.5.1. □

Remark 3.5.2. For any $\alpha \in (0, 1)$, we keep the same choice of ℓ, ρ as in (3.3.11) and choose δ as follows :

$$\delta = \left(\ln\frac{\kappa}{H}\right)^{-\frac{\alpha}{4}}\,. \tag{3.5.13}$$

This choice and Assumption (3.1.7) permit us to have the assumptions of Proposition 3.5.1 satisfied and make the error terms in its statement of order $o\left(\kappa H \ln\frac{\kappa}{H}\right)$. We have as $\kappa \longrightarrow +\infty$,

$$\frac{\kappa}{\ell\kappa H\ln\frac{\kappa}{H}} = \frac{\kappa^{\frac{1}{4}}}{H^{\frac{3}{4}}\ln\frac{\kappa}{H}} \ll 1\,,$$

$$\frac{\delta^{-1}(\kappa H)^2\ell^4}{\kappa H\ln\frac{\kappa}{H}} = \frac{1}{\left(\ln\frac{\kappa}{H}\right)^{1-\frac{\alpha}{4}}} \ll 1\,,$$

$$\frac{\delta^{-1}\kappa H\ln\frac{C_0\kappa}{H}\ell^{2\alpha}}{\kappa H\ln\frac{\kappa}{H}} = \frac{\ln\frac{C_0\kappa}{H}}{\left(\ln\frac{\kappa}{H}\right)^{1-\frac{\alpha}{4}}(\kappa H)^{\frac{\alpha}{2}}} \ll 1\,,$$

$$\ell(\kappa H)^{\frac{1}{2}}\rho^{\frac{1}{2}} = (\kappa H)^{\frac{1}{12}} \gg 1\,.$$

Remark 3.5.3. As a byproduct of the proof, we get also a useful estimate. Using the bound $|\psi|\leq 1$, it results from (3.5.5) :

$$\frac{(1-\delta)}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi,\sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0)+\nabla\varphi,Q_\ell(x_0)) \leq \frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi,\mathbf{A},Q_\ell(x_0)) + C\delta^{-1}(\kappa H)^2(\ell^4+\lambda^2\ell^{2\alpha})\,. \quad (3.5.14)$$

Using (3.5.3) and choosing ℓ, ρ as in (3.3.11) and δ as in (3.5.13), we get a function $\hat{r}:(0,+\infty)\longmapsto(0,+\infty)$ satisfying $\lim\limits_{t\longrightarrow+\infty}\hat{r}(t)=0$ and

$$\mathcal{E}_0(\psi,\sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0)+\nabla\varphi,Q_\ell(x_0)) \leq \mathcal{E}_0(\psi,\mathbf{A},Q_\ell(x_0)) + \ell^2\kappa H\ln\frac{\kappa}{H}\,\hat{r}(\kappa)\,, \quad (3.5.15)$$

for any $\widetilde{x}_0$ in $Q_\ell(x_0)$.

The next theorem presents the lower bound of the local energy in the domain $\mathcal{D}$ such that $\overline{\mathcal{D}}\subset\Omega$ and we deduce the lower bound of the global energy by replacing $\mathcal{D}$ with Ω.

Theorem 3.5.4. *Under Assumption* (3.1.2)*, if $H(\kappa)$ satisfies* (3.1.7)*, $(\psi,\mathbf{A})\in H^1(\Omega,\mathbb{C})\times H^1_{\rm div}(\Omega)$ is a minimizer of* (3.1.1) *and $\overline{\mathcal{D}}\subset\Omega$ is open, then,*

$$\mathcal{E}(\psi,\mathbf{A};\mathcal{D}) \geq \kappa^2\int_{\mathcal{D}}\hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)dx + o\left(\kappa H\ln\frac{\kappa}{H}\right)\,, \qquad \text{as}\,\kappa\longrightarrow+\infty\,.$$

Proof. The proof is similar to the one of Theorem 3.3.3.
Let

$$\mathcal{D}_{\ell,\rho} = \operatorname{int}\left(\cup_{\gamma\in\mathcal{I}_{\ell,\rho}}\overline{Q_{\gamma,\ell}}\right) \quad (3.5.16)$$

and

$$\overline{B}_{\gamma,\ell} = \sup_{x\in Q_{\gamma,\ell}}|B_0(x)|\,, \quad (3.5.17)$$

where $\mathcal{I}_{\ell,\rho}$ was introduced in (3.3.15).
If $(\psi,\mathbf{A})$ is a minimizer of (3.1.1), we have

$$\mathcal{E}(\psi,\mathbf{A};\mathcal{D}) = \mathcal{E}_0(\psi,\mathbf{A};\mathcal{D}_{\ell,\rho}) + \mathcal{E}_0(\psi,\mathbf{A};\mathcal{D}\setminus\mathcal{D}_{\ell,\rho}) + (\kappa H)^2\int_\Omega|\operatorname{curl}\mathbf{A}-B_0|^2\,dx\,,$$

where $\mathcal{E}_0(\psi,\mathbf{A};\mathcal{D})$ is introduced in (3.1.16).
Since the magnetic energy term and the energy in $\mathcal{D}\setminus\mathcal{D}_{\ell,\rho}$ are positive, we may write,

$$\mathcal{E}(\psi,\mathbf{A};\mathcal{D}) \geq \mathcal{E}_0(\psi,\mathbf{A};\mathcal{D}_{\ell,\rho})\,. \quad (3.5.18)$$

To estimate $\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{D}_{\ell,\rho})$, we notice that,

$$\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{D}_{\ell,\rho}) = \sum_{\gamma \in \mathcal{I}_{\ell,\rho}} \mathcal{E}_0(\psi, \mathbf{A}; Q_{\gamma,\ell}) .$$

Recall that for any $\widetilde{\gamma} \in \overline{Q_{\gamma,\ell}}$ we have $B_0(\widetilde{\gamma})$ satisfies (3.3.3). Then, we select $\widetilde{\gamma}$ such that

$$|B_0(\widetilde{\gamma})| = \overline{B}_{\gamma,\ell} .$$

Using (3.5.12), similarly as we did in the upper bound we recognize the upper Riemann sum, and get

$$\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{D}_{\ell,\rho}) \geq \kappa^2(1-\delta) \int_{\mathcal{D}_{\ell,\rho}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx - C\left(\frac{\kappa}{\ell} + \delta^{-1}(\kappa H)^2\ell^4 + \delta^{-1}\kappa^2 H^2 \lambda^2 \ell^{2\alpha}\right) . \tag{3.5.19}$$

Notice that using the regularity of $\partial\mathcal{D}$ and (3.1.2), there exists $C > 0$ such that

$$\forall \ell \in (0,1),\ \forall \rho \in (0,1), \quad |\mathcal{D} \setminus \mathcal{D}_{\ell,\rho}| \leq C(\ell + \rho) . \tag{3.5.20}$$

We get by using property (3) of f in Theorem 3.2.1, Assumption (3.1.7) and for some choice of ρ to be determined later

$$\begin{aligned}\int_{\mathcal{D}_{\ell,\rho}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx &\geq \int_{\mathcal{D}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx - \int_{\mathcal{D}\setminus\mathcal{D}_{\ell,\rho}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx \\ &\geq \int_{\mathcal{D}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx - C\frac{H}{\kappa}|\mathcal{D} \setminus \mathcal{D}_{\ell,\rho}| .\end{aligned} \tag{3.5.21}$$

This implies that

$$\mathrm{E_g}(\kappa, H) \geq \kappa^2(1-\delta) \int_{\mathcal{D}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx - r'(\kappa, H, \ell) , \tag{3.5.22}$$

where

$$r'(\kappa, H, \ell) = \mathcal{O}\left(\kappa H \ell + \kappa H \rho + \frac{\kappa}{\ell} + \delta^{-1}(\kappa H)^2 \ell^4 + \delta^{-1}\kappa^2 H^2 \lambda^2 \ell^{2\alpha}\right) .$$

Having in mind (3.5.3), then, the remainder term becomes

$$r'(\kappa, H, \ell) = \mathcal{O}\left(\kappa H \ell + \kappa H \rho + \frac{\kappa}{\ell} + \delta^{-1}(\kappa H)^2 \ell^4 + \delta^{-1}\kappa H \ln\frac{C_0\kappa}{H}\ell^{2\alpha}\right) .$$

The choice of the parameters δ in (3.5.13) and ρ, ℓ in (3.3.11) implies all error terms to be of lower order compared with $\kappa H \ln\frac{\kappa}{H}$. This finishes the proof of Theorem 3.5.4. $\square$

Remark 3.5.5. Notice that $\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{D}) \geq \mathcal{E}_0(\psi, \mathbf{A}; \mathcal{D}_{\ell,\rho})$. Using (3.5.19) and (3.5.21) with the same choices of δ, ρ and ℓ as in Remark 3.3.2, we obtain

$$\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{D}) \geq \kappa^2 \int_{\mathcal{D}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right) dx + o\left(\kappa H \ln\frac{\kappa}{H}\right) . \tag{3.5.23}$$

Moreover, we can replace $\mathcal{D}$ by Ω and get

$$\mathcal{E}_0(\psi,\mathbf{A};\Omega) \geq \kappa^2 \int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx + o\left(\kappa H \ln\frac{\kappa}{H}\right). \tag{3.5.24}$$

Proof of Corollary 3.1.4

Having in mind (3.1.16), we write

$$\mathcal{E}(\psi,\mathbf{A};\Omega) = \mathcal{E}_0(\psi,\mathbf{A};\Omega) + (\kappa H)^2 \int_\Omega |\operatorname{curl}\mathbf{A} - B_0|^2\,dx\,.$$

Using the estimate of $\mathcal{E}(\psi,\mathbf{A};\Omega)$ in Theorem 3.1.1, we get, as $\kappa \longrightarrow +\infty$

$$\mathcal{E}_0(\psi,\mathbf{A};\Omega) + (\kappa H)^2 \int_\Omega |\operatorname{curl}\mathbf{A} - B_0|^2\,dx \leq \kappa^2 \int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx + o\left(\kappa H \ln\frac{\kappa}{H}\right). \tag{3.5.25}$$

Remark 3.5.5 tells us that

$$\mathcal{E}_0(\psi,\mathbf{A};\Omega) \geq \kappa^2 \int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx + o\left(\kappa H \ln\frac{\kappa}{H}\right).$$

Therefore, (3.5.25) becomes

$$\begin{aligned}\kappa^2 \int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx + o\left(\kappa H \ln\frac{\kappa}{H}\right) + (\kappa H)^2 \int_\Omega |\operatorname{curl}\mathbf{A} - B_0|^2\,dx \\ \leq \kappa^2 \int_\Omega \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)\,dx + o\left(\kappa H \ln\frac{\kappa}{H}\right).\end{aligned} \tag{3.5.26}$$

By simplification, we get (3.1.15).

3.6 Proof of Theorem 3.1.5 : upper bound

One aim of this section is to derive a sharp estimate of $\mathcal{E}_0(\psi,\mathbf{A};Q_\ell(x_0))$, when $(\psi,\mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ is a minimizer of (3.1.1).

The proof of the next proposition is similar to that in [5, Proposition 6.2], by replacing $\dfrac{1}{R}$ by $\dfrac{b^{\frac12}}{R}$.

Proposition 3.6.1. *For $\alpha \in (0,1)$, there exist positive constants C and κ_0 such that if $\kappa \geq \kappa_0$, $\ell \in (0,\frac12)$, $\delta \in (0,1)$, $\rho > 0$, $\ell^2\kappa H\rho \geq 1$, $(\psi,\mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ is a minimizer of* (3.1.1), *and $(\ell, x_0, \widetilde{x}_0)$ a ρ-admissible triple, then,*

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi,\mathbf{A};Q_\ell(x_0)) \leq (1+\delta)\kappa^2 \hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right) + C\left(\frac{\kappa}{\ell} + \delta^{-1}\ell^4\kappa^2H^2 + \delta^{-1}\kappa^2H^2\lambda^2\ell^{2\alpha}\right), \tag{3.6.1}$$

where λ is introduced in (3.5.2).

Remark 3.6.2. Under Assumption (3.1.7), with the choices of ℓ, ρ in (3.3.11) and δ in (3.5.13), we get that the error terms in (3.6.1) are of order $\kappa H\, \ln\frac{\kappa}{H}$

Proposition 3.6.3. *For any $\alpha \in (0,1)$, there exist positive constants $\widehat{C}_\alpha$ and κ_0 such that if $\kappa \geq \kappa_0$, $H(\kappa)$ satisfies (3.1.7), ℓ is chosen as in (3.3.11), δ as in (3.5.13), $\ell^2\kappa H\rho \geq 1$, $(\psi, \mathbf{A})$ is a minimizer of (3.1.1), and $(\ell, x_0, \widetilde{x}_0)$ a ρ-admissible triple, then*

$$\left|\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi, \sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0) + \nabla(\varphi_{x_0,\widetilde{x}_0} + \phi_{x_0}), Q_\ell(x_0)) - \kappa^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)\right| \leq \widehat{C}_\alpha\, \kappa H \left(\ln\frac{\kappa}{H}\right)^{\frac{\alpha}{4}}\,, \quad (3.6.2)$$

where $\mathbf{A}_0$ is the magnetic potential introduced in (3.2.2), σ_ℓ denotes the sign of B_0, ϕ_{x_0} is defined in (3.5.1) and $\varphi_{x_0,\widetilde{x}_0}$ is the function satisfying (3.3.3).

Proof.
Lower bound : We refer to (3.5.11) and (3.5.6). We obtain

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi, \sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0) + \nabla(\varphi_{x_0,\widetilde{x}_0} + \phi_{x_0}), Q_\ell(x_0)) \geq \kappa^2\hat{f}\left(\frac{H|B_0(\widetilde{x}_0)|}{\kappa}\right) - C\frac{\kappa}{\ell}\,, \quad (3.6.3)$$

where C is a positive constant.
If we select ℓ as in (3.3.11), we get

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi, \sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0) + \nabla(\varphi_{x_0,\widetilde{x}_0} + \phi_{x_0}), Q_\ell(x_0)) \geq \kappa^2\hat{f}\left(\frac{H|B_0(\widetilde{x}_0)|}{\kappa}\right) - C\left(\kappa^5 H\right)^{\frac{1}{4}}\,. \quad (3.6.4)$$

Assumption (3.1.7) permits to verify that the remainder $(\kappa^5 H)^{\frac{1}{4}} = \mathcal{O}(\kappa H(\ln\frac{\kappa}{H})^{\frac{\alpha}{4}})$.
Upper bound : Collecting (3.5.14) and (3.6.1), we get for any $\alpha \in (0,1)$, the existence of $C' > 0$ such that

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi, \sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0) + \nabla(\varphi_{x_0,\widetilde{x}_0} + \phi_{x_0}), Q_\ell(x_0)) \leq \kappa^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right) + C'\left(\frac{\kappa}{\ell} + \delta^{-1}\ell^4\kappa^2H^2 + \delta^{-1}\kappa^2H^2\lambda^2\ell^{2\alpha}\right)\,, \quad (3.6.5)$$

where λ is introduced in (3.5.2).
Using (3.5.3) and selecting ℓ as in (3.3.11) and δ as in (3.5.13), we get the existence of a constant C_α such that

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(\psi, \sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0) + \nabla(\varphi_{x_0,\widetilde{x}_0} + \phi_{x_0}), Q_\ell(x_0)) \leq \kappa^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right) + C_\alpha\kappa H\left(\ln\frac{\kappa}{H}\right)^{\frac{\alpha}{4}}\,. \quad (3.6.6)$$

This achieves the proof of the lemma. □

The next lemma will be useful in the proof of Theorem 3.1.5.

Lemma 3.6.4. *For any $C_1 > 0$, there exist positive constants C and κ_0 such that if $\ell \in (0,1)$,*

$\kappa_0 \leq \kappa$ and $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\rm div}(\Omega)$ is a solution of (3.1.5), then

$$\int_{\mathcal{V}_\ell(\Gamma,C_1)} |(\nabla - i\kappa H\mathbf{A})\psi|^2\,dx \leq C\kappa^2\ell\left(1+\frac{1}{\kappa\ell^{\frac{3}{2}}}\right), \tag{3.6.7}$$

where

$$\mathcal{V}_\ell(\Gamma,C_1) = \left\{x\in\Omega : \mathrm{dist}(x,\Gamma) \leq C_1\ell \ \textit{and}\ d(x,\partial\Omega) \geq \frac{\ell}{C_1}\right\}.$$

Proof. Using (3.4.5) and the fact that the range of $\hat{f}$ is the interval $[0,1/2]$, we get

$$\|(\nabla - i\kappa H\mathbf{A})\psi\|_{L^2(\Omega)} \leq C\kappa\,. \tag{3.6.8}$$

Hence the improvment given by the lemma is when $\frac{1}{C}\kappa^{-2} \leq \ell \leq \ell_0$.
Let $C_2 > C_1$ and for ℓ small enough we define the following sets $\mathcal{D}^1_\ell = \mathcal{V}_\ell(\Gamma,C_1)$ and $\mathcal{D}^2_\ell = \mathcal{V}_\ell(\Gamma,C_2)$. We can construct a cut-off function $\chi_\ell \in C^\infty_c(\Omega)$ such that

$$0 \leq \chi_\ell \leq 1 \text{ in } \mathbb{R}^2\,,\quad \mathrm{supp}\chi_\ell \subset \mathcal{D}^2_\ell \subset\subset \Omega\,,\quad \chi_\ell = 1 \text{ in } \mathcal{D}^1_\ell \quad\text{and}\quad |\nabla\chi_\ell| \leq \frac{C}{\ell} \text{ in } \mathbb{R}^2\,, \tag{3.6.9}$$

where C is a positive constant independent of ℓ.
The minimizer ψ satisfies

$$\kappa^2\psi(1-|\psi|^2) = -(\nabla - i\kappa H\mathbf{A})^2\psi \quad \text{in}\,\Omega\,. \tag{3.6.10}$$

We multiply the above equation by $\chi_\ell\bar{\psi}$, it results from an integration by parts that

$$\begin{aligned}\kappa^2\int_{\mathcal{D}^2_\ell}\chi_\ell(1-|\psi|^2)|\psi|^2\,dx &= \int_{\mathcal{D}^2_\ell}(\nabla - i\kappa H\mathbf{A})\psi\;\overline{\chi_\ell(\nabla - i\kappa H\mathbf{A})\psi + \psi\nabla\chi_\ell}\,dx\\ &= \int_{\mathcal{D}^2_\ell}\chi_\ell\,|(\nabla - i\kappa H\mathbf{A})\psi|^2\,dx + \int_{\mathcal{D}^2_\ell}\nabla\chi_\ell\,\bar{\psi}\,(\nabla - i\kappa H\mathbf{A})\psi\,dx\,.\end{aligned} \tag{3.6.11}$$

Using Hölder inequality, we have

$$\left|\int_{\mathcal{D}^2_\ell}\nabla\chi_\ell\,\bar{\psi}\,(\nabla - i\kappa H\mathbf{A})\psi\,dx\right| \leq \|(\nabla - i\kappa H\mathbf{A})\psi\|_{L^2(\Omega,\mathbb{C}^2)}\,\||\nabla\chi_\ell|\,\psi\|_{L^2(\mathcal{D}^2_\ell)}\,. \tag{3.6.12}$$

Notice that $|\mathcal{D}^2_\ell| \leq C'\ell$. Using (3.6.9) and the bound $\|\psi\|_\infty \leq 1$, we obtain

$$\||\nabla\chi_\ell|\,\psi\|_{L^2(\mathcal{D}^2_\ell)} \leq C''\ell^{-\frac{1}{2}}\,. \tag{3.6.13}$$

Putting (3.6.8) and (3.6.13) into (3.6.12), we get

$$\left|\int_{\mathcal{D}^2_\ell}\nabla\chi_\ell\,\bar{\psi}(\nabla - i\kappa H\mathbf{A})\psi\,dx\right| \leq C\,\kappa\,\ell^{-\frac{1}{2}}\,, \tag{3.6.14}$$

and consequently

$$\int_{\mathcal{D}_\ell^2} \chi_\ell |(\nabla - i\kappa H\mathbf{A})\psi|^2\,dx \le \kappa^2 \int_{\mathcal{D}_\ell^2} \chi_\ell (1-|\psi|^2)|\psi|^2\,dx + C\,\kappa\,\ell^{-\frac{1}{2}}\,. \tag{3.6.15}$$

The lemma easily follows from the control of the area of $\mathcal{D}_\ell^2$ and from observing that $\chi_\ell = 1$ on $\mathcal{D}_\ell^1$. □

Remark 3.6.5. We get a similar estimate by replacing in the lemma Γ by the boundary $\partial\mathcal{D}$ of a regular open set $\mathcal{D}$ compactly contained in Ω.

End of the proof of Theorem 3.1.5.
The proof of (3.1.18) is already obtained in Theorem 3.5.4. Hence it remains only to give the proof of (3.1.20).
We keep the same notation as in (3.3.14), (3.3.15) and (3.5.16). If $(\psi,\mathbf{A})$ is a minimizer of (3.1.1), we start with (3.1.17) and write,

$$\mathcal{E}(\psi,\mathbf{A};\mathcal{D}) = \mathcal{E}_0(\psi,\mathbf{A};\mathcal{D}_{\ell,\rho}) + \mathcal{E}_0(\psi,\mathbf{A};\mathcal{D}\setminus\mathcal{D}_{\ell,\rho}) + (\kappa H)^2\int_\Omega |\operatorname{curl}\big(\mathbf{A}-\mathbf{F}\big)|^2\,dx\,. \tag{3.6.16}$$

To estimate $\mathcal{E}_0(\psi,\mathbf{A};\mathcal{D}_{\ell,\rho})$, we notice that,

$$\mathcal{E}_0(\psi,\mathbf{A};\mathcal{D}_{\ell,\rho}) = \sum_{\gamma\in\mathcal{I}_{\ell,\rho}} \mathcal{E}_0(\psi,\mathbf{A};Q_{\gamma,\ell})\,.$$

Remark 3.6.2 tells us that the error terms in (3.6.1) are of order $\kappa H\ \ln\frac{\kappa}{H}$. Therfore, using (3.6.1), we get

$$\mathcal{E}_0(\psi,\mathbf{A};\mathcal{D}_{\ell,\rho}) \le \kappa^2 \sum_{\gamma\in\mathcal{I}_{\ell,\rho}} \hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{\gamma})|\right)\ell^2 + o\left(\kappa H\ln\frac{\kappa}{H}\right)\,,\qquad \text{as } \kappa\longrightarrow+\infty\,.$$

We select $\widetilde{\gamma}\in\overline{Q_\ell(\gamma)}$ such that $|B_0(\widetilde{\gamma})| = \underline{B}_{\gamma,\ell}$, where $\underline{B}_{\gamma,\ell}$ is defined in (3.3.14). By monotonicity of $\hat{f}$, $\hat{f}$ is Riemann-integrable and its integral is larger than any lower Riemann sum. Thus

$$\mathcal{E}_0(\psi,\mathbf{A};\mathcal{D}_{\ell,\rho}) \le \kappa^2 \int_{\mathcal{D}_{\ell,\rho}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)dx + o\left(\kappa H\ln\frac{\kappa}{H}\right)\,,\qquad \text{as } \kappa\longrightarrow+\infty\,. \tag{3.6.17}$$

Moreover, recalling that $\hat{f}$ is a positive function and $\mathcal{D}_{\ell,\rho}\subset\mathcal{D}$, (3.6.17) becomes

$$\mathcal{E}_0(\psi,\mathbf{A};\mathcal{D}_{\ell,\rho}) \le \kappa^2 \int_{\mathcal{D}} \hat{f}\left(\frac{H}{\kappa}|B_0(x)|\right)dx + o\left(\kappa H\ln\frac{\kappa}{H}\right)\,,\qquad \text{as } \kappa\longrightarrow+\infty\,. \tag{3.6.18}$$

For estimating $\mathcal{E}_0(\psi,\mathbf{A};\mathcal{D}\setminus\mathcal{D}_{\ell,\rho})$, we use Lemma 3.6.4, Remark 3.6.5 and we keep the same choice of ℓ and ρ as in (3.3.11), which implies $\rho\ll\ell$, we obtain that

$$\int_{\mathcal{D}\setminus\mathcal{D}_{\ell,\rho}} |(\nabla - i\kappa H\mathbf{A})\psi|^2\,dx \le C(\,\kappa^2\,\ell + \,\kappa\,\ell^{-\frac{1}{2}})\,. \tag{3.6.19}$$

Adding the second term in the energy leads to

$$\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{D} \setminus \mathcal{D}_{\ell,\rho}) \leq C(\,\kappa^2\,\ell + \,\kappa\,\ell^{-\frac{1}{2}}\,)\,. \tag{3.6.20}$$

The second term in the right hand side is controlled by the first one if

$$\kappa \ell^{\frac{3}{2}} \gg 1\,.$$

This is effectively satisfied with our choice of ℓ and the condition on $H(\kappa)$.
In order to obtain the term $\kappa^2 \ell$ in (3.6.20) comparatively small with $\kappa H \ln \frac{\kappa}{H}$, we need a stronger condition than (3.1.7) on $H(\kappa)$. In fact, we have

$$\frac{\kappa^2 \ell}{\kappa H \ln \frac{\kappa}{H}} = \left(\frac{\kappa^3}{H^5}\right)^{\frac{1}{4}} \frac{1}{\ln \frac{\kappa}{H}}\,,$$

and thanks to (3.1.19), as $\kappa \longrightarrow +\infty$,

$$\frac{1}{\ln \frac{\kappa}{H}} \ll 1 \quad \text{and} \quad \frac{\kappa^3}{H^5} \leq C\,,$$

where C is a positive constant.
This implies that

$$\kappa^2 \ell = o\left(\kappa H \ln \frac{\kappa}{H}\right)\,,$$

and consequently

$$\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{D} \setminus \mathcal{D}_{\ell,\rho}) = o\left(\kappa H \ln \frac{\kappa}{H}\right)\,. \tag{3.6.21}$$

Corollary 3.1.4 tells us that, under Assumption (3.1.7),

$$(\kappa H)^2 \int_\Omega |\operatorname{curl} \mathbf{A} - B_0|^2\,dx = o\left(\kappa H \ln \frac{\kappa}{H}\right)\,, \qquad \text{as } \kappa \longrightarrow +\infty\,. \tag{3.6.22}$$

Therefore, by collecting (3.6.18), (3.6.21) and (3.6.22) and inserting then into (3.6.16), we finish the proof of (3.1.20).

3.7 Vortices

This section is devoted to the proof of Theorem 3.1.6. We keep the choice of ℓ given in (3.3.11) :

$$\ell = (\kappa H)^{-\frac{1}{4}}\,,$$

but we select ρ and α as follows :

$$\rho = \left(\ln \frac{\kappa}{H}\right)^{-\frac{1}{2}}\,, \qquad \alpha = \frac{1}{2}\,. \tag{3.7.1}$$

3.7.1 Energy in a ρ-admissible box

If $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\mathrm{div}}(\Omega;\mathbb{R})$, we consider for all ρ-admissible pair (ℓ, x_0) the local energy in $Q_\ell(x_0)$:

$$\mathcal{E}_0(\psi, \mathbf{A}, Q_\ell(x_0)) = \int_{Q_\ell(x_0)} \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 + \frac{\kappa^2}{2}(1-|\psi|^2)^2 \right) dx\,.$$

3.7.2 Division of the square $Q_\ell(x_0)$

Let $H = H(\kappa)$ be a function satisfying (3.1.7). For reasons that will become clear in Proposition 3.7.3, we need to divide $Q_\ell(x_0)$ into $\mathcal{N} = M^2$ disjoint open squares $(Q^j_{\delta(\kappa)})_{j\in\mathcal{J}}$ such that

$$Q_\ell(x_0) = \cup_{j\in\mathcal{J}} \overline{Q^j_{\delta(\kappa)}}\,,$$

with

$$M = \left[2^{\frac{7}{8}} \left(\kappa H\right)^{\frac{1}{4}} \left(\ln \frac{\kappa}{H}\right)^{-\frac{7}{8}} \right]\,, \tag{3.7.2}$$

where for $t \in \mathbb{R}$, $[t]$ denotes the integer part of t.
The side length of theses squares is consequently

$$\delta(\kappa) = \frac{\ell}{M} \sim 2^{-\frac{7}{8}}(\kappa H)^{-\frac{1}{2}} \left(\ln \frac{\kappa}{H}\right)^{\frac{7}{8}}\,. \tag{3.7.3}$$

Let us introduced for all ρ-admissible triple $(\ell, x_0, \widetilde{x}_0)$ the functions b and R by

$$R(\kappa, H, \widetilde{x}_0) = 2^{-\frac{7}{8}} \left(\ln \frac{\kappa}{H}\right)^{\frac{7}{8}} |B_0(\widetilde{x}_0)|^{\frac{1}{2}} \quad \text{and} \quad b(\kappa, H, \widetilde{x}_0) = \frac{H}{\kappa}|B_0(\widetilde{x}_0)|\,. \tag{3.7.4}$$

Notice that $b(\kappa, H, \widetilde{x}_0)$ and $\dfrac{1}{R(\kappa, H, \widetilde{x}_0)}$ are *uniformly* $o(1)$ as $\kappa \longrightarrow +\infty$, in the following sense : For all $\epsilon > 0$ there exists $\kappa_0 > 0$ such that $\forall \kappa \geq \kappa_0$, H satisfying (3.1.7), ρ introduced in (3.7.1) and any ρ-admissible triple $(\ell, x_0, \widetilde{x}_0)$

$$|b(\kappa, H, \widetilde{x}_0)| + \frac{1}{R(\kappa, H, \widetilde{x}_0)} < \epsilon\,.$$

In fact, we have as $\kappa \longrightarrow +\infty$

$$R(\kappa, H, \widetilde{x}_0) \geq 2^{-\frac{7}{8}} \left(\ln \frac{\kappa}{H}\right)^{\frac{7}{8}} \rho^{\frac{1}{2}} \geq \frac{1}{C} \left(\ln \frac{\kappa}{H}\right)^{\frac{5}{8}} \gg 1\,.$$

Since $B_0 \in C^\infty(\overline{\Omega})$, we have also

$$0 < b(\kappa, H, \widetilde{x}_0) \leq \frac{H}{\kappa}\overline{\beta}_0 \ll 1\,, \tag{3.7.5}$$

where

$$\overline{\beta}_0 = \sup_{x\in\overline{\Omega}} |B_0(x)| \,. \tag{3.7.6}$$

More precisely, let

$$\hat{b}(\kappa, H, \beta) = b(\kappa, H, \widetilde{x}_0) \qquad \text{and} \qquad \hat{R}(\kappa, H, \beta) = R(\kappa, H, \widetilde{x}_0)\,, \tag{3.7.7}$$

where $\beta = |B_0(\widetilde{x}_0)|$.
We define the function :

$$h(\kappa, H) = \max\left(\left(\ln\frac{\kappa}{H}\right)^{-\frac{3}{8}}, \sup_{\overline{\beta}_0\geq\beta\geq\left(\ln\frac{\kappa}{H}\right)^{-\frac{1}{2}}} \operatorname{err}(\hat{b}(\kappa,H,\beta),\hat{R}(\kappa,H,\beta)) \right), \tag{3.7.8}$$

where $\operatorname{err}(b, R)$ is defined in Proposition 3.2.13.
Notice that h satisfies

$$h(\kappa, H) = o(1)\,, \qquad \text{as } \kappa \longrightarrow +\infty\,. \tag{3.7.9}$$

Next, we will use a method introduced by E. Sandier and S. Serfaty in [46]. We distinguish in the family indexed by $\mathcal{J}$ two types of squares respectively called the '*nice squares*' $(Q^j_{\delta(\kappa)})$ which are indexed in $\mathcal{J}^n$ and the '*bad squares*' $(Q^j_{\delta(\kappa)})$ indexed in $\mathcal{J}^b$. The set $\mathcal{J}^n$ is the set of indices $j \in \mathcal{J}$ such that

$$\mathcal{E}_0(\psi, \sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0)+\nabla\varphi, Q^j_{\delta(\kappa)}) \leq \delta(\kappa)^2\kappa^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)(1+\,h(\kappa,H)^{\frac{1}{2}}). \tag{3.7.10}$$

The set $\mathcal{J}^b$ is the set of indices $j \in \mathcal{J}$ such that

$$\mathcal{E}_0(\psi, \sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0)+\nabla\varphi, Q^j_{\delta(\kappa)}) > \delta(\kappa)^2\kappa^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)(1+h(\kappa,H)^{\frac{1}{2}}). \tag{3.7.11}$$

Hence we have $\mathcal{J} = \mathcal{J}^n \cup \mathcal{J}^b$. We denote by $\mathcal{N}^g$ the cardinal of $\mathcal{J}^n$ and by $\mathcal{N}^b$ the cardinal of $\mathcal{J}^b$.

Lemma 3.7.1. *There exist positive constants C and κ_0 such that if $\kappa \geq \kappa_0$, then*

$$\mathcal{N}^b \leq C\,\frac{h(\kappa,H)^{\frac{1}{2}}}{1-h(\kappa,H)^{\frac{1}{2}}}\mathcal{N}^n\,, \tag{3.7.12}$$

where h is introduced in (3.7.8).

Proof. Recall that $\mathbf{A}_0$ is the magnetic potential introduced in (3.2.2), ϕ_{x_0} is defined in (3.5.1) and that, for $\widetilde{x}_0 \in \overline{Q_\ell(x_0)}$, $\varphi_{x_0,\widetilde{x}_0}$ is the function satisfying (3.3.3).
Having in mind the definition of b and R in (3.7.4) and their properties, and using (3.2.15), we

get from (3.5.8) and (3.5.9) the following inequality

$$\begin{aligned}\mathcal{E}_0(\psi, \sigma_\ell |B_0(\widetilde{x}_0)| \mathbf{A}_0(x - x_0) + \nabla\varphi, Q^j_{\delta(\kappa)}) &\geq \frac{e_N(b,R)}{b} \\ &\geq \frac{R^2}{b} \hat{f}(b)\,(1 - \mathrm{err}(b,R))\,, \end{aligned} \tag{3.7.13}$$

where $\varphi = \phi_{x_0} + \varphi_{x_0,\widetilde{x}_0}$, e_N is introduced in (3.2.3), $b = b(\kappa, H, \widetilde{x}_0)$ and $R = R(\kappa, H, \widetilde{x}_0)$. As a consequence of (3.7.8), (3.7.13) becomes

$$\mathcal{E}_0(\psi, \sigma_\ell |B_0(\widetilde{x}_0)| \mathbf{A}_0(x - x_0) + \nabla\varphi, Q^j_{\delta(\kappa)}) \geq \kappa^2 \delta(\kappa)^2 \hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)(1 - \ h(\kappa, H))\,. \tag{3.7.14}$$

Notice that

$$|Q_\ell(x_0)| = \sum_{j\in\mathcal{J}} \left|Q^j_{\delta(\kappa)}\right| = (\mathcal{N}^n + \mathcal{N}^b)\,\delta(\kappa)^2\,. \tag{3.7.15}$$

Thus, we may write

$$\sum_{j\in\mathcal{J}} \mathcal{E}_0\left(\psi, \sigma_\ell |B_0(\widetilde{x}_0)| \mathbf{A}_0(x - x_0) + \nabla\varphi, Q^j_{\delta(\kappa)}\right) = \mathcal{E}_0(\psi, \sigma_\ell |B_0(\widetilde{x}_0)| \mathbf{A}_0(x - x_0) + \nabla\varphi, Q_\ell(x_0))\,. \tag{3.7.16}$$

When (3.1.7) is satisfied, we have uniformly

$$\ell^2 \kappa H \rho = (\kappa H)^{\frac{1}{2}} \left(\ln \frac{\kappa}{H}\right)^{-\frac{1}{2}} \gg 1\,, \qquad \text{as } \kappa \longrightarrow +\infty\,.$$

Hence the assumptions of Proposition 3.6.3 are satisfied. Putting (3.7.16) into (3.6.2), using (3.7.1) and (3.7.15), we get

$$\begin{aligned}&\left|\sum_{j\in\mathcal{J}} \left[\mathcal{E}_0\left(\psi, \sigma_\ell |B_0(\widetilde{x}_0)| \mathbf{A}_0(x - x_0) + \nabla\varphi, Q^j_{\delta(\kappa)}\right) - \kappa^2 \delta(\kappa)^2 \hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)\right]\right| \\ &\qquad\leq C\,(\mathcal{N}^b + \mathcal{N}^n)\,\delta(\kappa)^2\,\kappa H \left(\ln \frac{\kappa}{H}\right)^{\frac{1}{8}}\,. \end{aligned} \tag{3.7.17}$$

Using the monotonicity of $\hat{f}$ and remembering that $(\ell, x_0, \widetilde{x}_0)$ is a ρ-admissible triple, we get,

$$\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right) \geq \hat{f}\left(\frac{H}{\kappa}\rho\right)\,. \tag{3.7.18}$$

Using (3.2.11), (3.7.1) and (3.7.5), we obtain as $\kappa \longrightarrow +\infty$

$$\kappa^2\,\hat{f}\left(\frac{H}{\kappa}\rho\right) \geq \kappa H \rho \left(\ln \frac{\kappa}{H\rho}\right)\left(1 + \hat{s}\left(\frac{H}{\kappa}\rho\right)\right) \tag{3.7.19}$$

$$\geq \kappa H \rho \left(\ln \frac{\kappa}{H} + \ln \frac{1}{\rho}\right)\left(1 + \hat{s}\left(\frac{H}{\kappa}\rho\right)\right) \tag{3.7.20}$$

$$\geq \kappa H \left(\ln \frac{\kappa}{H}\right)^{\frac{1}{2}} (1 + \hat{s}_1(\kappa, H))\,, \tag{3.7.21}$$

where $\hat{s}_1(\kappa,H)=\hat{s}\left(\frac{H}{\kappa}\left(\ln\frac{\kappa}{H}\right)^{-\frac{1}{2}}\right)$ is uniformly $o(1)$ for H satisfying (3.1.7).
Collecting (3.7.18)-(3.7.19), we get for κ sufficiently large

$$\begin{aligned}\kappa^2\,\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right) &\geq \kappa H\,\left(\ln\frac{\kappa}{H}\right)^{\frac{1}{2}}(1+\hat{s}_1(\kappa,H))\\ &\geq \frac{\kappa H}{2}\,\left(\ln\frac{\kappa}{H}\right)^{\frac{1}{2}}\,.\end{aligned}$$

Multiplying both sides by $\left(\ln\frac{\kappa}{H}\right)^{-\frac{3}{8}}$ and using the definition of h in (3.7.8), we get

$$\kappa^2\,\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)\,h(\kappa,H)\geq\frac{\kappa H}{2}\,\left(\ln\frac{\kappa}{H}\right)^{\frac{1}{8}}\,. \tag{3.7.22}$$

Puting (3.7.22) into (3.7.17), we obtain

$$\begin{aligned}&\left|\sum_{j\in\mathcal{J}}\left[\mathcal{E}_0\left(\psi,\sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0)+\nabla\varphi,Q^j_{\delta(\kappa)}\right)-\kappa^2\delta(\kappa)^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)\right]\right|\\ &\qquad\leq C\,(\mathcal{N}^b+\mathcal{N}^n)\,\kappa^2\,\delta(\kappa)^2\,\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)\,h(\kappa,H)\,.\end{aligned} \tag{3.7.23}$$

Using (3.7.11), (3.7.14) and (3.7.23), we may write

$$\begin{aligned}&\mathcal{N}^b\,\kappa^2\,\delta(\kappa)^2\,\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)\,h(\kappa,H)^{\frac{1}{2}}\\ &\leq\sum_{j\in\mathcal{J}^b}\left[\mathcal{E}_0\left(\psi,\sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0)+\nabla\varphi,Q^j_{\delta(\kappa)}\right)-\kappa^2\delta(\kappa)^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)\right]\\ &\leq\sum_{j\in\mathcal{J}}\left[\mathcal{E}_0\left(\psi,\sigma_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0)+\nabla\varphi,Q^j_{\delta(\kappa)}\right)-\kappa^2\delta(\kappa)^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)\right]\\ &\qquad\qquad+\sum_{j\in\mathcal{J}^n}\kappa^2\delta(\kappa)^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)\,h(\kappa,H)\\ &\leq C\,(\mathcal{N}^b+\mathcal{N}^n)\kappa^2\delta(\kappa)^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)h(\kappa,H)+\mathcal{N}^n\kappa^2\delta(\kappa)^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)h(\kappa,H).\end{aligned}$$

We divide both sides by $\kappa^2\delta(\kappa)^2\hat{f}\left(\frac{H}{\kappa}|B_0(\widetilde{x}_0)|\right)\,h(\kappa,H)^{\frac{1}{2}}$ to get (3.7.12). □

Remark 3.7.2. Using (3.7.9) and (3.1.7), we obtain uniformly

$$\mathcal{N}^b\ll\mathcal{N}^n\,,\qquad\text{as }\kappa\longrightarrow+\infty\,. \tag{3.7.24}$$

More precisely, we mean that $\mathcal{N}^b=\mathcal{N}^n\,e(\kappa,H,\ell,\widetilde{x}_0,x_0)$, with $e(\kappa,H,\ell,\widetilde{x}_0,x_0)$ is uniformly $o(1)$ for any $\kappa\geq\kappa_0$, any ρ-admissible triple $(\ell,\widetilde{x}_0,x_0)$, any H satisfying (3.1.7).

3.7.3 The results of Sandier-Serfaty.

Now we recall an important result of Sandier-Serfaty [46]. Define the energy of $(u,A) \in H^1(\mathcal{D};\mathbb{C}) \times H^1_{\rm div}(\mathcal{D};\mathbb{R})$ in a domain $\mathcal{D} \subset \mathbb{R}^2$ as follows

$$J_{\mathcal{D}}(u,A) = \int_{\mathcal{D}} |(\nabla - iA)u|^2 + \frac{\kappa^2}{2}(1-|u|^2)^2 + |\operatorname{curl} A - h_{\rm ex}|^2\, dx\,. \qquad (3.7.25)$$

The next proposition is essentially proved[3] in [46, Proposition 5.1].

Proposition 3.7.3. *Let $\hat{h} : (0,+\infty) \longrightarrow (0,+\infty)$ such that $\lim_{t\longrightarrow+\infty} \hat{h}(t) = 0$, there exist two functions s_1, $s_2 : (0,+\infty) \longrightarrow (0,+\infty)$ satisfying*

$$\lim_{t\longrightarrow+\infty} s_1(t) = 0\,, \qquad \lim_{t\longrightarrow+\infty} s_2(t) = 0\,. \qquad (3.7.26)$$

Assume that $h_{\rm ex}$ is a function of κ and K is a square of side length $\gamma(\kappa)$ such that

$$|\ln\kappa| \ll h_{\rm ex} \ll \kappa^2 \qquad \text{and} \qquad \ln\frac{\kappa}{\sqrt{h_{\rm ex}}} \ll h_{\rm ex}\gamma(\kappa)^2 \ll \min\left(h_{\rm ex}, \left(\ln\frac{\kappa}{\sqrt{h_{\rm ex}}}\right)^2\right)\,, \quad \text{as } \kappa \longrightarrow \infty\,. \qquad (3.7.27)$$

If $(u,A) \in C^1(\overline{K};\mathbb{C}) \times C^1(\overline{K};\mathbb{R}^2)$ verifies

$$J_K(u,A) \leq h_{\rm ex}\gamma(\kappa)^2 \ln\frac{\kappa}{\sqrt{h_{\rm ex}}}(1+\hat{h}(\kappa))\,, \qquad (3.7.28)$$

then, there exist disjoint disks $(D(a_i,r_i))_{i=1}^m$ such that :

1. $\sum r_i \leq h_{\rm ex}^{-\frac{1}{2}}$
2. $|u| > \frac{1}{2}$ *on* $\cup_i \partial D(a_i,r_i)$
3. *If* $d_i = \deg\left(\frac{u}{|u|}, \partial D(a_i,r_i)\right)$, *then, as* $\kappa \longrightarrow +\infty$

$$2\pi\sum_{i=1}^m d_i \geq h_{\rm ex}\gamma(\kappa)^2(1-s_1(\kappa)) \qquad \text{and} \qquad 2\pi\sum_{i=1}^m |d_i| \leq h_{\rm ex}\gamma(\kappa)^2(1+s_2(\kappa))\,. \qquad (3.7.29)$$

We will present a proof of Proposition 3.7.3 in Appendix A.

The next lemma will give us that $\delta(\kappa)$, the side length of the square $Q^j_{\delta(\kappa)}$, satisfies (3.7.27) and will be useful in Proposition 3.7.5.

Lemma 3.7.4. *Under the assumptions of the previous subsection we have*

$$\delta(\kappa)^2 = \frac{1}{\varepsilon_1(\kappa,\ell,x_0,\widetilde{x}_0,H)}\,\frac{1}{\kappa H\,|B_0(\widetilde{x}_0)|}\ln\frac{\kappa}{H\,|B_0(\widetilde{x}_0)|}\,, \qquad \text{as } \kappa \longrightarrow +\infty \qquad (3.7.30)$$

$$\delta(\kappa)^2 = \varepsilon_2(\kappa,\ell,x_0,\widetilde{x}_0,H)\,\frac{1}{\kappa H\,|B_0(\widetilde{x}_0)|}\left(\ln\frac{\kappa}{H\,|B_0(\widetilde{x}_0)|}\right)^2\,, \quad \text{as } \kappa \longrightarrow +\infty\,, \qquad (3.7.31)$$

where $\varepsilon_1(\kappa,\ell,x_0,\widetilde{x}_0,H)$ and $\varepsilon_2(\kappa,\ell,x_0,\widetilde{x}_0,H)$ are uniformly $o(1)$ as $\kappa \longrightarrow +\infty$.

[3] We replaced ε by $\frac{1}{\kappa}$. We can indeed verify that only the upper bound of $J_K(u,A)$ is needed with no additional condition on $f(\varepsilon)$ and that the $o(1)$ are actually uniformity under uniform assumptions. Note also that we do not use in this proposition that (u,A) is a critical point of J_Ω.

Proof.

Proof of (3.7.30). We know that for all ρ-admissible triple $(\ell, x_0, \widetilde{x}_0)$

$$\frac{1}{\kappa H|B_0(\widetilde{x}_0)|}\ln\frac{\kappa}{H|B_0(\widetilde{x}_0)|}\leq\frac{1}{\kappa H\rho}\ln\frac{\kappa}{H\rho}\leq\frac{3}{2}\frac{1}{\kappa H}\left(\ln\frac{\kappa}{H}\right)^{\frac{3}{2}},$$

Also, we know that

$$\frac{1}{\kappa H}\left(\ln\frac{\kappa}{H}\right)^{\frac{3}{2}}\ll 2^{-\frac{7}{4}}\frac{1}{\kappa H}\left(\ln\frac{\kappa}{H}\right)^{\frac{7}{4}}\sim\delta(\kappa)^2\,.$$

Proof of (3.7.31). On the other hand, we have

$$\frac{1}{\kappa H|B_0(\widetilde{x}_0)|}\left(\ln\frac{\kappa}{H|B_0(\widetilde{x}_0)|}\right)^2\geq\frac{1}{\kappa H\overline{\beta}_0}\left(\ln\frac{\kappa}{H\overline{\beta}_0}\right)^2\geq C\frac{1}{\kappa H}\left(\ln\frac{\kappa}{H}\right)^2,$$

where C is a positive constant and $\overline{\beta}_0$ is introduced in (3.7.6).
It is clear that

$$\frac{1}{\kappa H}\left(\ln\frac{\kappa}{H}\right)^2\gg 2^{-\frac{7}{4}}\frac{1}{\kappa H}\left(\ln\frac{\kappa}{H}\right)^{\frac{7}{4}}\sim\delta(\kappa)^2\,.$$

□

We can prove the following result regarding the vortices of the minimizers in the '*nice squares*'. We start with the admissible squares contained in $\Omega\cap\{B_0>0\}$.

Proposition 3.7.5. *Under Assumptions* (3.1.2) *and* (3.1.7) *there exists* s_1, $s_2:(0,+\infty)\longrightarrow(0,+\infty)$ *two functions satisfying* (3.7.26) *and such that, for any* $(\ell, x_0, \widetilde{x}_0)$ *such that* $\overline{Q^j_{\delta(\kappa)}}\subset\Omega\cap\{B_0>\rho\}$ *and* $\widetilde{x}_0\in\overline{Q^j_{\delta(\kappa)}}$ *for which* $Q^j_{\delta(\kappa)}$ *is a nice square, and any minimizer* $(\psi,\mathbf{A})\in H^1(\Omega,\mathbb{C})\times H^1_{\rm div}(\Omega,\mathbb{R}^2)$ *of* (3.1.1), *there exist disjoint disks* $(D(a_{i,j},r_{i,j}))_{i=1}^{m_j}$ *in* $Q^j_{\delta(\kappa)}$ *such that*

- $\sum_{i=1}^{m_j} r_{i,j}\leq(\kappa HB_0(\widetilde{x}_0))^{-\frac{1}{2}}$
- $|\psi|>\frac{1}{2}$ *on* $\cup_j\,\partial D(a_{i,j},r_{i,j})$ (3.7.32)
- *If* $d_{i,j}$ *is the winding number of* $\frac{\psi}{|\psi|}$ *restricted to* $\partial D(a_{i,j},r_{i,j})$, *then*

$$2\pi\sum_{i=1}^{m_j}d_{i,j}\geq\delta(\kappa)^2\kappa HB_0(\widetilde{x}_0)(1-s_1(\kappa))\,,\qquad\textit{as }\kappa\longrightarrow+\infty\,,\tag{3.7.33}$$

and

$$2\pi\sum_{i=1}^{m_j}|d_{i,j}|\leq\delta(\kappa)^2\kappa HB_0(\widetilde{x}_0)(1+s_2(\kappa))\,,\qquad\textit{as }\kappa\longrightarrow+\infty\,.\tag{3.7.34}$$

Proof.
We will apply Proposition 3.7.3 with

$$K=Q^j_{\delta(\kappa)},\ \gamma(\kappa)=\delta(\kappa),\ h_{\rm ex}=\kappa HB_0(\widetilde{x}_0),\ u=e^{-i\kappa H\varphi}\psi\text{ and }A(x)=\kappa H\,B_0(\widetilde{x}_0)\,\mathbf{A}_0(x-x_0)\,,\tag{3.7.35}$$

where $\mathbf{A}_0$ is the magnetic potential introduced in (3.2.2) and $\varphi=\phi_{x_0}+\varphi_{x_0,\widetilde{x}_0}$, with ϕ_{x_0} defined in (3.5.1) and $\varphi_{x_0,\widetilde{x}_0}$ in (3.3.3).

Let us verify that the conditions of the proposition are satisfied for this choice.
First, we start by proving (3.7.28). Since $\operatorname{curl} \mathbf{A}_0 = 1$, then,

$$\operatorname{curl} A = \kappa H \, B_0(\widetilde{x}_0) = h_{\rm ex}\,,$$

and consequently

$$\int_{Q^j_{\delta(\kappa)}} |\operatorname{curl} A - h_{\rm ex}|^2 \, dx = 0\,. \tag{3.7.36}$$

This implies that, for any $j \in \mathcal{J}^n$

$$\begin{aligned} J_K(u,A) &= \int_{Q^j_{\delta(\kappa)}} \left(|(\nabla - i\kappa H(B_0(\widetilde{x}_0)\mathbf{A}_0(x-x_0) + \nabla\varphi))\psi|^2 + \frac{\kappa^2}{2}\left(1-|\psi|^2\right)^2 \right) dx \\ &= \mathcal{E}_0(\psi, B_0(\widetilde{x}_0)\mathbf{A}_0(x-x_0)+\nabla\varphi, Q^j_{\delta(\kappa)})\,. \end{aligned} \tag{3.7.37}$$

Since $Q^j_{\delta(\kappa)}$ is a nice square, then,

$$J_K(u,A) \leq \delta(\kappa)^2\,\kappa^2 \hat{f}\left(\frac{H}{\kappa}\,B_0(\widetilde{x}_0)\right)(1+h(\kappa,H)^{\frac12})\,, \qquad \text{as } \kappa \longrightarrow +\infty\,. \tag{3.7.38}$$

As a consequence of (3.2.11), (3.7.38) becomes

$$J_K(u,A) \leq \frac12\,\delta(\kappa)^2\kappa H B_0(\widetilde{x}_0)\,\ln\frac{\kappa}{HB_0(\widetilde{x}_0)}(1+\hat{h}(\kappa,H))\,, \qquad \text{as } \kappa \longrightarrow +\infty\,, \tag{3.7.39}$$

where

$$\hat{h}(\kappa,H) = h(\kappa,H)^{\frac12} + \hat{s}\left(\frac{H}{\kappa}\right) + \hat{s}\left(\frac{H}{\kappa}\right)\,h(\kappa,H)^{\frac12}\,.$$

Notice that the function $\hat{h}(\kappa,H)$ is uniformly $o(1)$ as $\kappa \longrightarrow +\infty$ and H satisfying (3.1.7).
Secondly we prove (3.7.27). In fact, under Assumption (3.1.7), we can easly prove, uniformly as $\kappa \longrightarrow +\infty$,

$$|\ln\kappa| \ll h_{\rm ex} \ll \kappa^2\,,$$

and

$$\min\left(h_{\rm ex}, \left(\ln\frac{\kappa}{\sqrt{h_{\rm ex}}}\right)^2\right) = \left(\ln\frac{\kappa}{\sqrt{h_{\rm ex}}}\right)^2\,.$$

Thanks to Lemma 3.7.4, we get that (3.7.27) is satisfied and in this way we achieve the proof of Proposition 3.7.5. □

In light of Lemma 3.7.1, we deduce from Proposition 3.7.5 the distribution of vortices in a ρ-admissible square Q_ℓ.

Proposition 3.7.6. *Suppose that Assumptions* (3.1.2) *and* (3.1.7) *are true. There exists two functions* $s_1,\, s_2 : (0;+\infty) \longrightarrow (0;+\infty)$ *satisfying* (3.7.26) *and the following is true. Let* $(\psi,\mathbf{A}) \in H^1(\Omega,\mathbb{C}) \times H^1_{\rm div}(\Omega,\mathbb{R}^2)$ *be a minimizer of* (3.1.1) *and* (ℓ, x_0) *such that* $\overline{Q_\ell(x_0)} \subset \Omega \cap \{B_0 > \rho\}$.

There exist a family of disjoint disks (indexed by $\mathcal{K}=\mathcal{K}_{\ell,x_0}$) $(D(\widetilde{a}_k,\widetilde{r}_k))_{k\in\mathcal{K}}$ in $Q_\ell(x_0)$ such that

- $$\sum_{k\in\mathcal{K}}\widetilde{r}_k \leq (\kappa H B_0(\widetilde{x}_0))^{-\frac{1}{2}}\left(\frac{\ell}{\delta(\kappa)}\right)^2(1+o(1))\,,\qquad \text{as } \kappa\longrightarrow+\infty \tag{3.7.40}$$
- $$|\psi|>\frac{1}{2}\quad \textit{on } \cup_k\partial D(\widetilde{a}_k,\widetilde{r}_k) \tag{3.7.41}$$
- *If $\widetilde{d}_k$ is the winding number of $\dfrac{\psi}{|\psi|}$ restricted to $\partial D(\widetilde{a}_k,\widetilde{r}_k)$, then as $\kappa\longrightarrow+\infty$*

$$2\pi\sum_{k\in\mathcal{K}}\widetilde{d}_k \geq \ell^2\kappa H\,B_0(\widetilde{x}_0)\,(1-s_1(\kappa))\quad\text{and}\quad 2\pi\sum_{k\in\mathcal{K}}|\widetilde{d}_k|\leq \ell^2\kappa H\,B_0(\widetilde{x}_0)\,(1+\,s_2(\kappa))\,. \tag{3.7.42}$$

Here, the function $o(1)$ is bounded independently of the choice of $\widetilde{x}_0$ and the minimizer $(\psi,\mathbf{A})$.

Proof. Recall that $Q_\ell(x_0)$ is decomposed into $\mathcal{N}^n$ '*nice squares*' $(Q^j_{\delta(k)})_{j\in\mathcal{J}^n}$ and $\mathcal{N}^b$ '*bad squares*' $(Q^j_{\delta(k)})_{j\in\mathcal{J}^b}$.
In every nice square $Q^j_{\delta(\kappa)}$, Proposition 3.7.5 tells us that there exist disjoint disks $(D(a_{i,j},r_{i,j}))_{i=1}^{m_j}$ such that (3.7.32), (3.7.33) and (3.7.34) hold. Let $(D(\widetilde{a}_k,\widetilde{r}_k))_{k\in\mathcal{K}}=(D(a_{i,j},r_{i,j}))_{i,j}$ be the family of disjoint disks in $\cup_{j\in\mathcal{J}^n}Q^j_{\delta(\kappa)}$. Clearly

$$\sum_{k\in\mathcal{K}}\widetilde{r}_k=\sum_{j\in\mathcal{J}^n}\sum_{i=1}^{m_j}r_{i,j}\,.$$

This implies that

$$\sum_{k\in\mathcal{K}}\widetilde{r}_k\leq(\kappa H B_0(\widetilde{x}_0))^{-\frac{1}{2}}\mathcal{N}^n\,.$$

Having in mind (3.7.15) and (3.7.24) , we have, as $\kappa\longrightarrow+\infty$,

$$\mathcal{N}^n\,\frac{\delta(\kappa)^2}{\ell^2}\longrightarrow 1\,. \tag{3.7.43}$$

This implies that

$$\sum_{k\in\mathcal{K}}\widetilde{r}_k\leq(\kappa H B_0(\widetilde{x}_0))^{-\frac{1}{2}}\left(\frac{\ell}{\delta(\kappa)}\right)^2(1+o(1))\,,$$

where the function $o(1)$ is bounded independently of the choice of $\widetilde{x}_0$.
Let $\widetilde{d}_k$ be the winding number of $\frac{\psi}{|\psi|}$ restricted to $\partial D(\widetilde{a}_k,\widetilde{r}_k)$, then, for any $k\in\mathcal{K}$

$$\sum_{k\in\mathcal{K}}\widetilde{d}_k=\sum_{j\in\mathcal{J}^n}\sum_{i=1}^{m_j}d_{i,j}\,.$$

Since from (3.7.33), (3.7.34) and (3.7.43), we get, as $\kappa \longrightarrow +\infty$,

$$2\pi \sum_{k\in\mathcal{K}} \widetilde{d}_k = 2\pi \sum_{j\in\mathcal{J}^n} \sum_{i=1}^{m_j} d_{i,j} \geq \mathcal{N}^n \delta(\kappa)^2 \kappa H B_0(\widetilde{x}_0)\,(1 - s_1(\kappa))$$
$$\geq \ell^2 \kappa H B_0(\widetilde{x}_0)\,(1 - s_1(\kappa))\ , \tag{3.7.44}$$

and

$$2\pi \sum_{k\in\mathcal{K}} |\widetilde{d}_k| = 2\pi \sum_{j\in\mathcal{J}^n} \sum_{i=1}^{m_j} |d_{i,j}| \leq \mathcal{N}^n \delta(\kappa)^2 \kappa H B_0(\widetilde{x}_0)\,(1 + s_2(\kappa))$$
$$\leq \ell^2 \kappa H B_0(\widetilde{x}_0)\,(1 + s_2(\kappa))\ . \tag{3.7.45}$$

This finishes the proof of Proposition 3.7.6. □

3.7.4 Proof of Theorem 3.1.6

Let $(\psi, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega, \mathbb{R}^2)$ be a minimizer of (3.1.1) and $\Gamma_\ell := \ell\mathbb{Z} \times \ell\mathbb{Z}$ a lattice of $\mathbb{R}^2$. For all $\gamma \in \Gamma_\ell$, we consider the family of squares $Q_\ell(\gamma)$ and $\widetilde{\gamma} \in Q_\ell(\gamma)$. Consider an open set $S \subset \Omega \cap \{B_0 > 0\}$ such that the boundary of S is smooth. Let

$$\mathcal{J}_\ell = \{\gamma\,;\quad \overline{Q_\ell(\gamma)} \subset S \cap \{B_0 > \rho\}\,\}\,, \tag{3.7.46}$$

$$\mathcal{M} = \mathrm{card}\,\mathcal{J}_\ell\,, \tag{3.7.47}$$

and

$$S_\ell = \mathrm{int}\,\left(\cup_{\gamma\in\mathcal{J}_\ell} \overline{Q_\ell(\gamma)}\right)\,. \tag{3.7.48}$$

Then, as $\kappa \longrightarrow +\infty$, we have

$$\mathcal{M} \times \ell^2 \longrightarrow |S|\,. \tag{3.7.49}$$

Proof of (3) **:**
Proposition 3.7.6 tells us that there exist disjoint disks $(D(\widetilde{a}_{k,\gamma}, \widetilde{r}_{k,\gamma}))_{k\in\mathcal{K}_{\ell,\gamma}}$ in each square $Q_\ell(\gamma)$ with $\overline{Q_\ell(\gamma)} \subset \Omega \cap \{B_0 > \rho\}$ such that (3.7.40) and (3.7.42) hold. We introduce the measure by

$$\mu_\kappa := \frac{2\pi}{\kappa H} \sum_{\gamma\in\mathcal{J}_\ell} \sum_{k\in\mathcal{K}_{\ell,\gamma}} \widetilde{d}_{k,\gamma}\, \delta_{\widetilde{a}_{k,\gamma}}\,, \tag{3.7.50}$$

where $\widetilde{d}_{k,\gamma}$ is the winding number introduced before (3.7.42) and $\delta_{\widetilde{a}_{k,\gamma}}$ is the unit Dirac mass at $\widetilde{a}_{k,\gamma}$.
Having in mind (3.7.42) we have for any $(\ell, \gamma, \widetilde{\gamma})$ such that $\overline{Q_\ell(\gamma)} \subset \Omega \cap \{B_0 > \rho\}$ and $\widetilde{\gamma} \in \overline{Q_\ell(\gamma)}$

$$B_0(\widetilde{\gamma})\ell^2(1 - s_1(\kappa)) \leq \frac{2\pi}{\kappa H} \sum_{k\in\mathcal{K}_{\ell,\gamma}} \widetilde{d}_{k,\gamma} \leq B_0(\widetilde{\gamma})\ell^2(1 + s_2(\kappa))\,.$$

Using (3.7.49), we obtain

$$\left(\sum_{\gamma\in\mathcal{J}_\ell} B_0(\widetilde{\gamma})\ell^2\right) - 2\,\overline{\beta}_0|S|\,s_1(\kappa) \leq \frac{2\pi}{\kappa H}\sum_{\gamma\in\mathcal{J}_\ell}\sum_{k\in\mathcal{K}_{\ell,\gamma}} \widetilde{d}_{k,\gamma} \leq \left(\sum_{\gamma\in\mathcal{J}_\ell} B_0(\widetilde{\gamma})\ell^2\right) + 2\,\overline{\beta}_0|S|\,s_2(\kappa)\,. \tag{3.7.51}$$

Here, we have used the fact that $B_0(\widetilde{\gamma}) \leq \overline{\beta}_0$ to estimate the errors terms, where $\overline{\beta}_0$ is introduced in (3.7.6).

Now it is time to determine $\sum_{\gamma\in\mathcal{J}_\ell} B_0(\widetilde{\gamma})\ell^2$. We will do this in two steps :

Upper bound : Notice that till now $\widetilde{\gamma}$ was an arbitrary point in $Q_\ell(\gamma)$, but that our estimates are independent of this choice. We now select $\widetilde{\gamma} \in \overline{Q_\ell(\gamma)}$ such that $B_0(\widetilde{\gamma}) = \underline{B}_{\gamma,\ell}$ with $\underline{B}_{\gamma,\ell}$ satisfying (3.3.14), and get

$$\sum_{\gamma\in\mathcal{J}_\ell} B_0(\widetilde{\gamma})\ell^2 = \sum_{\gamma\in\mathcal{J}_\ell} \underline{B}_{\gamma,\ell}\ell^2\,.$$

We recognize in the right hand side above the lower Riemann sum of $x \longrightarrow B_0(x)$ and we use that $S_\ell \subset S$ to obtain

$$\sum_{\gamma\in\mathcal{J}_\ell} B_0(\widetilde{\gamma})\ell^2 \leq \int_S B_0(x)\,dx\,. \tag{3.7.52}$$

Lower bound : We select $\widetilde{\gamma} \in Q_\ell(\gamma)$ such that $B_0(\widetilde{\gamma}) = \overline{B}_{\gamma,\ell}$ with $\overline{B}_{\gamma,\ell}$ satisfies (3.5.17). Similarly to what we did in the upper bound above, we get

$$\sum_{\gamma\in\mathcal{J}_\ell} B_0(\widetilde{\gamma})\ell^2 \geq \int_{S_\ell} B_0(x)\,dx\,.$$

Notice that using the regularity of ∂S and (3.1.2), we have as $\kappa \longrightarrow +\infty$

$$|S\setminus S_\ell| = \mathcal{O}(\ell|\partial S|)\,. \tag{3.7.53}$$

Therefore

$$\begin{aligned}\int_{S_\ell} B_0(x)\,dx &= \int_S B_0(x)\,dx - \int_{S\setminus S_\ell} B_0(x)\,dx\\ &\geq \int_S B_0(x)\,dx - \overline{\beta}_0|S\setminus S_\ell|\\ &\geq \int_S B_0(x)\,dx - C\,\ell\,,\end{aligned}$$

where $\overline{\beta}_0$ is introduced in (3.7.6) and C is a positive constant.

This implies that

$$\sum_{\gamma\in\mathcal{J}_\ell} B_0(\widetilde{\gamma})\ell^2 \geq \int_S B_0(x)\,dx - C\,\ell\,. \tag{3.7.54}$$

The estimates in (3.7.52) and (3.7.54) allow us to deduce from (3.7.51) that

$$-\,C\,\ell - 2\,\overline{\beta}_0|S|\,s_1(\kappa) \leq \mu_\kappa(S) - \int_S B_0(x)\,dx \leq +2\,\overline{\beta}_0|S|\,s_2(\kappa)\,. \tag{3.7.55}$$

Consequently, as $\kappa \longrightarrow +\infty$

$$\mu_\kappa(S) \longrightarrow \int_S B_0(x)\,dx\,, \qquad \forall S \subset \Omega \cap \{B_0 > 0\}\,. \tag{3.7.56}$$

In light of (3.7.56), we can easily show that μ_κ converge weakly to $\mu = B_0(x)\,dx$, which means that :

$$\mu_\kappa(f) \longrightarrow \mu(f)\,, \qquad \forall f \in C_0(\Omega \cap \{B_0 > 0\})\,.$$

Proof of (1) **:** We will prove that the sum of the radii of the disks $(D(\widetilde{a}_{k,\gamma}, \widetilde{r}_{k,\gamma}))_{k\in\mathcal{K}_{\ell,\gamma}\,\gamma\in\mathcal{J}_\ell}$ is less than

$$(\kappa H)^{\frac{1}{2}} \left(\ln \frac{\kappa}{H}\right)^{-\frac{7}{4}} \int_S \frac{1}{\sqrt{B_0(x)}}\,dx\,(1 + o(1))\,.$$

In fact, remembering the choice of $\delta(\kappa)$ in (3.7.3), (3.7.49) and that $\overline{Q_\ell(\gamma)} \subset \Omega \cap \{B_0 > 0\}$, we have

$$\begin{aligned}\sum_{\gamma\in\mathcal{J}_\ell} \widetilde{r}_{k,\gamma} &= \sum_{\gamma\in\mathcal{J}_\ell}\sum_{k\in\mathcal{K}} \widetilde{r}_k \\ &\leq (\kappa H)^{\frac{1}{2}} \left(\ln \frac{\kappa}{H}\right)^{-\frac{7}{4}} \sum_{\gamma\in\mathcal{J}_\ell} \frac{1}{\sqrt{B_0(\widetilde{\gamma})}}\,\ell^2\,(1 + o(1))\,.\end{aligned} \tag{3.7.57}$$

We select $\widetilde{\gamma} \in \overline{Q_\ell(\gamma)}$ such that

$$\frac{1}{\sqrt{B_0(\widetilde{\gamma})}} = \inf_{\widehat{\gamma}\in Q_{\gamma,\ell}} \frac{1}{\sqrt{B_0(\widehat{\gamma})}}\,,$$

and we recognize in the right hand side of (3.7.57) the lower Riemann sum of $x \longrightarrow \frac{1}{\sqrt{B_0(x)}}$, we get

$$\sum_{\gamma\in\mathcal{J}_\ell} \widetilde{r}_{k,\gamma} \leq (\kappa H)^{\frac{1}{2}} \left(\ln \frac{\kappa}{H}\right)^{-\frac{7}{4}} \int_S \frac{1}{\sqrt{B_0(x)}}\,dx\,(1 + o(1))\,.$$

End of the proof of Theorem 3.1.6 : In $\{B_0 < 0\} \cap \Omega$, we apply Proposition 3.7.3 with

$$K = Q^j_{\delta(\kappa)},\ \gamma(\kappa) = \delta(\kappa),\ h_{\rm ex} = -\kappa H B_0(\widetilde{x}_0),\ u = e^{i\kappa H\varphi}\overline{\psi} \text{ and } A(x) = -\kappa H\, B_0(\widetilde{x}_0)\,\mathbf{A}_0(x - x_0)\,. \tag{3.7.58}$$

So we get that, the convergence of mesure μ_κ in (3.7.56) is still true when $S \subset \Omega \cap \{B_0 < 0\}$. Similarly, we can control the convergence of $|\mu_\kappa|(S)$. Now we observe that the support of μ_κ does not meet $\{B_0 = 0\}$. Hence $\mu_\kappa(S) = \mu_\kappa(S \cap \{B_0 < 0\}) + \mu_\kappa(S \cap \{B_0 > 0\})$ and we can apply the previous arguments to $S_- = S \cap \{B_0 < 0\}$ and $S_+ = S \cap \{B_0 > 0\}$.

Chapitre 4

Pinning with a variable magnetic field of the two dimensional Ginzburg-Landau model

We study the Ginzburg-Landau energy of a superconductor with a variable magnetic field and a pinning term in a bounded smooth two dimensional domain Ω. Supposing that the Ginzburg-Landau parameter and the intensity of the magnetic field are large and of the same order, we determine an accurate asymptotic formula for the minimizing energy. This asymptotic formula displays the influence of the pinning term. Also, we discuss the existence of non-trivial solutions and prove some asymptotics of the third critical field.

4.1 Introduction

We consider a bounded, open and simply connected set $\Omega \subset \mathbb{R}^2$ with smooth boundary. We suppose that Ω models an inhomogeneous superconducting sample submitted to an applied external magnetic field. The energy of the sample is given by the so called pinned Ginzburg-Landau functional,

$$\mathcal{E}_{\kappa,H,a,B_0}(\psi,\mathbf{A}) = \int_\Omega \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 + \frac{\kappa^2}{2}(a(x,\kappa) - |\psi|^2)^2 \right) dx + \kappa^2 H^2 \int_\Omega |\operatorname{curl}\mathbf{A} - B_0|^2 \, dx \,. \tag{4.1.1}$$

Here κ and H are two positive parameters such that κ describes the properties of the material, and H measures the variation of the intensity of the applied magnetic field. The modulus $|\psi|^2$ of the wave function (order parameter) $\psi \in H^1(\Omega;\mathbb{C})$ measures the density of the superconducting electron Cooper pairs. The magnetic potential $\mathbf{A}$ belongs to $H^1_{\mathrm{div}}(\Omega)$ where

$$H^1_{\mathrm{div}}(\Omega) = \{\mathbf{A} = (\mathbf{A}_1,\mathbf{A}_2) \in H^1(\Omega)^2 \ : \ \operatorname{div}\mathbf{A} = 0 \text{ in } \Omega\,,\ \mathbf{A}\cdot\nu = 0 \text{ on } \partial\Omega\,\}\,, \tag{4.1.2}$$

with ν being the unit interior normal vector of $\partial\Omega$.
The function $\kappa H \operatorname{curl}\mathbf{A}$ gives the induced magnetic field.

When $\psi \equiv 0$ and $(\psi, \mathbf{A})$ is a minimizer or a critical point of the functional, we call this pair normal state. In our case it is easy to see normal minimizers (if any) are necessarily in the form $(0, \mathbf{A})$ with $\mathbf{A}$ in $H^1_{\mathrm{div}}(\Omega)$ such that $\operatorname{curl} \mathbf{A} = B_0$. This solution is unique and denoted by $\mathbf{F}$. A natural question will be to determine under which condition this normal solution is a minimizer.

The function $B_0 \in C^\infty(\overline{\Omega})$ is the intensity of the external magnetic field which is variable in our problem. Let

$$\Gamma = \{x \in \overline{\Omega} : B_0(x) = 0\} . \tag{4.1.3}$$

We assume that either Γ is empty or that B_0 satisfies :

$$\begin{cases} |B_0| + |\nabla B_0| > 0 & \text{in } \overline{\Omega} \\ \nabla B_0 \times \vec{n} \neq 0 & \text{on } \Gamma \cap \partial\Omega . \end{cases} \tag{4.1.4}$$

The assumption in (4.1.4) implies that for any open set ω relatively compact in Ω, $\Gamma \cap \omega$ is either empty, or consists of a union of smooth curves.

The energy $\mathcal{E}_{\kappa,H,a,B_0}$ considered here is slightly different from the classical Ginzburg-Landau energy in the sense that there is a varying term denoted by $a(x,\kappa)$ penalizing the variations of the order parameter ψ and called the pinning term. This term arises also naturally in the microscopic derivation of the Ginzburg-Landau theory from BCS theory (see [21]) without any a priori assumption on the sign of a.

Assumption 4.1.1. *The function $a(x,\kappa)$ is real, defined on $\overline{\Omega} \times [\kappa_0, +\infty)$, and satisfies for some $\kappa_0 > 0$ the following assumptions :*

(A_1)

$$\forall \kappa \geq \kappa_0 \,, a(\cdot, \kappa) \in C^1(\overline{\Omega}) . \tag{4.1.5}$$

(A_2)

$$\sup_{x \in \overline{\Omega},\, \kappa \geq \kappa_0} |a(x,\kappa)| < +\infty . \tag{4.1.6}$$

(A_3)

$$\forall \kappa \geq \kappa_0 \,, \qquad \sup_{x \in \overline{\Omega}} |\nabla_x a(x,\kappa)| < +\infty . \tag{4.1.7}$$

(A_4) *There exists a positive constant C_1, such that,*

$$\forall \kappa \geq \kappa_0 \,, \qquad \mathcal{L}\left(\partial\{a(x,\kappa) > 0\}\right) \leq C_1 \, \kappa^{\frac{1}{2}} , \tag{4.1.8}$$

where $\mathcal{L}$ is the "length" of $\partial\{a(x,\kappa) > 0\}$ in Ω in a sense that will be explained in (4.3.1).

$$L(\kappa) = \sup_x |\nabla_x a(x,\kappa)| , \tag{4.1.9}$$

$$\overline{a} = \sup_{x \in \overline{\Omega},\, \kappa \geq \kappa_0} a(x,\kappa) \tag{4.1.10}$$

and

$$\underline{a} = \inf_{x \in \overline{\Omega},\, \kappa \geq \kappa_0} a(x,\kappa). \tag{4.1.11}$$

The assumption in (A_3) gives a uniform control for any κ of the oscillation of $a(.,\kappa)$ which will be made precise later by an assumption on $L(\kappa)$. Notice that the normal state $(0,\mathbf{F})$ is a critical point of the functional in (4.1.1). It is standard, starting from a minimizing sequence, to prove the existence of minimizers in $H^1(\Omega;\mathbb{C}) \times H^1_{\rm div}(\Omega)$ of the functional $\mathcal{E}_{\kappa,H,a,B_0}$. A minimizer $(\psi,\mathbf{A})$ of (4.1.1) is a weak solution of the Ginzburg-Landau equations,

$$\begin{cases} -(\nabla - i\kappa H\mathbf{A})^2\psi = \kappa^2\left(a(x,\kappa) - |\psi|^2\right)\psi & \text{in} \quad \Omega \quad (a)\\ -\nabla^\perp \operatorname{curl}(\mathbf{A}-\mathbf{F}) = \dfrac{1}{\kappa H}\operatorname{Im}(\overline{\psi}\,(\nabla - i\kappa H\mathbf{A})\psi) & \text{in} \quad \Omega \quad (b)\\ \nu\cdot(\nabla - i\kappa H\mathbf{A})\psi = 0 & \text{on} \quad \partial\Omega \quad (c)\\ \operatorname{curl}\mathbf{A} = \operatorname{curl}\mathbf{F} & \text{on} \quad \partial\Omega \quad (d)\,. \end{cases} \tag{4.1.12}$$

Here, $\operatorname{curl}\mathbf{A} = \partial_{x_1}\mathbf{A}_2 - \partial_{x_2}\mathbf{A}_1$ and $\nabla^\perp \operatorname{curl}\mathbf{A} = (\partial_{x_2}(\operatorname{curl}\mathbf{A}), -\partial_{x_1}(\operatorname{curl}\mathbf{A}))$.

Let us introduce the magnetic Schrödinger operator in an open set $\widetilde{\Omega}$ in $\mathbb{R}^2$:

$$P^{\widetilde{\Omega}}_{A,V} = -(\nabla - iA)^2 + V(x)\,, \tag{4.1.13}$$

where $A \in H^1_{\rm div}(\widetilde{\Omega})$ and V is a continuous function bounded from below.
The form domain of $P^{\widetilde{\Omega}}_{A,V}$ is

$$\mathcal{V}(\widetilde{\Omega}) = \{u \in L^2(\widetilde{\Omega})\,, \quad (\nabla - iA)u \in L^2(\widetilde{\Omega})\,, \quad (V+C)^{\frac{1}{2}}u \in L^2(\widetilde{\Omega})\}\,,$$

and its operator domain is given by

$$D(P^{\widetilde{\Omega}}_{A,V}) := \{u \in \mathcal{V}(\widetilde{\Omega})\,, \quad P^{\widetilde{\Omega}}_{A,V}u \in L^2(\widetilde{\Omega}), \quad \nu\cdot(\nabla - iA)u = 0 \text{ on } \partial\widetilde{\Omega}\}\,.$$

Then, $(4.1.12)_{a,c}$ reads

$$P^{\Omega}_{A,V}\,\psi = -\kappa^2\,|\psi|^2\psi\,,$$

with $A = \kappa H\mathbf{A}$, $\psi \in D(P^{\Omega}_{A,V})$ and $V = -\kappa^2\,a$.

There are many papers on the Ginzburg-Landau functional with a pinning term, most of them study the influence of the pinning term on the location of *vortices*, i.e. the zeros of the minimizing order parameter. For the functional without a magnetic field (i.e. $B_0 = 0$ in (4.1.1)), the influence of the pinning term is studied in [37] and more recently in [39] and the references therein. The pinning term (i.e. the function a) in [37] is a step function independent of κ ; more complicated κ-dependent periodic step functions are considered in [39]. The magnetic version of the functional in [37] is studied in [31, 33].

In [4], Aftalion, Sandier and Serfaty considered a **smooth** and κ-dependent pinning term a satisfying :

(H_1) $L(\kappa) \ll \kappa H$.

(H_2) There exist a continuous function $a(x)$, a positive constant a_0 and, for all $\kappa \geq 0$, there exist two functions $\sigma(\kappa) = o\left(\left(\ln\left|\ln\frac{1}{\kappa}\right|\right)^{-\frac{1}{2}}\right)$ and $\beta(x,\kappa) \geq 0$ such that,

$$\min_{B(x,\sigma(\kappa))} \beta(x,\kappa) = 0\,, \qquad a(x,\kappa) = a(x) + \beta(x,\kappa)\,, \qquad \text{and} \qquad 0 < a_0 \leq a(x) \leq 1\,.$$

The study contains the case when $a(x,\kappa) = a(x)$ ($\beta = 0$) but also cases with a κ- control of the x-oscillation of $\beta(\cdot,\kappa)$ which could increase with κ. In the scales of this paper, the results in [4] are valid when the parameter H is of order $\frac{|\ln\kappa|}{\kappa}$ as $\kappa \longrightarrow +\infty$.

Extending the discussion, the functional in (4.1.1) is close to models of Bose-Einstein condensates (see e.g. [1, 2]).

In this paper, we will analyze how the pinning term appears in the asymptotics of the energy in the presence of a strong external variable magnetic field (see Theorem 4.1.2 below). Also, we discuss the influence of the pinning on the asymptotic expression of the third critical field H_{C_3} (see Theorems 4.1.6 and 4.1.7).

We focus on the regime of large values of κ, $\kappa \to +\infty$ and we study the ground state energy defined as follows,

$$\mathrm{E_g}(\kappa, H, a, B_0) = \inf\left\{\mathcal{E}_{\kappa,H,a,B_0}(\psi,\mathbf{A}) \ : \ (\psi,\mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)\right\}. \tag{4.1.14}$$

More precisely, we give an asymptotic estimate which is valid in the simultaneous limit $\kappa \longrightarrow +\infty$ and $H(\kappa) \longrightarrow +\infty$ with the constraint that $\frac{H(\kappa)}{\kappa}$ remains asymptotically of uniform size, that is satisfying

$$\lambda_{\min} \leq \frac{H(\kappa)}{\kappa} \leq \lambda_{\max} \qquad (\kappa \geq \kappa_0)\,, \tag{4.1.15}$$

where $\lambda_{\min}$, $\lambda_{\max}$ are positive constants such that $\lambda_{\min} < \lambda_{\max}$.
The behavior of $\mathrm{E_g}(\kappa, H, a, B_0)$ involves a function $\hat{f} : [0,+\infty) \longrightarrow [0,\frac{1}{2}]$ introduced in [6, Theorem 2.1]. The function $\hat{f}$ is increasing, continuous and $\hat{f}(b) = \frac{1}{2}$, for all $b \geq 1$.

Theorem 4.1.2. *Suppose that Assumption 4.1.1 and* (4.1.15) *hold, and*

$$L(\kappa) = \mathcal{O}(\kappa^{\frac{1}{2}}) \qquad \text{as } \kappa \to +\infty\,. \tag{4.1.16}$$

The ground state energy in (4.1.14) *satisfies*

$$\begin{aligned}\mathrm{E_g}(\kappa, H, a, B_0) = \kappa^2 &\int_{\{a(x,\kappa)>0\}} a(x,\kappa)^2\,\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)dx \\ &+ \frac{\kappa^2}{2}\int_{\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\,dx + o\left(\kappa^2\right)\,, \qquad \text{as } \kappa \longrightarrow +\infty\,.\end{aligned} \tag{4.1.17}$$

When $\Omega \cap \{a(x,\kappa) > 0\} = \varnothing$, we obtain directly from (4.1.14)

$$\mathcal{E}_{\kappa,H,a,B_0}(\psi,\mathbf{A}) \geq \frac{\kappa^2}{2}\int_\Omega a(x,\kappa)^2\,dx = \mathcal{E}_{\kappa,H,a,B_0}(0,\mathbf{F})\,.$$

Hence the minimizer of $\mathcal{E}_{\kappa,H,a,B_0}$ is the normal state. In physical terms, this case corresponds to the case when we are above the critical temperature.

We will describe later cases when the remainder term in (4.1.17) is indeed small compared with the leading order term (see Section 4.6).

The assumptions in Theorem 4.1.2 contain the case when the function a is constant and equals 1, which was proved in [5] under Assumption (4.1.15).

Along the proof of Theorem 4.1.2, we obtain an estimate of the 'magnetic energy' as follows :

Corollary 4.1.3. *Under the assumptions of Theorem 4.1.2, we have*

$$(\kappa H)^2 \int_\Omega |\,\mathrm{curl}\,\mathbf{A} - B_0|^2\,dx = o(\kappa^2)\,, \qquad \text{as } \kappa \longrightarrow +\infty\,. \tag{4.1.18}$$

If $\mathcal{D}$ is a domain in Ω, we introduce the local energy in $\mathcal{D}$ of $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ by :

$$\mathcal{E}_0(\psi, \mathbf{A}; a, \mathcal{D}) = \int_{\mathcal{D}} |(\nabla - i\kappa H\mathbf{A})\psi|^2\,dx + \frac{\kappa^2}{2}\int_{\mathcal{D}} (a(x,\kappa) - |\psi|^2)^2\,dx\,. \tag{4.1.19}$$

The next theorem gives an estimate of the local energy $\mathcal{E}_0(\psi, \mathbf{A}; a, \mathcal{D})$.

Theorem 4.1.4. *Under the assumptions of Theorem 4.1.2, if $(\psi, \mathbf{A})$ is a minimizer of* (4.1.1) *and $\mathcal{D}$ is regular set such that $\overline{\mathcal{D}} \subset \Omega$, then*

$$\begin{aligned}\mathcal{E}_0(\psi, \mathbf{A}; a, \mathcal{D}) = \kappa^2 \int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} & a(x,\kappa)^2\,\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)\,dx \\ &+ \frac{\kappa^2}{2}\int_{\mathcal{D}\cap\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\,dx + o\left(\kappa^2\right)\,, \qquad \text{as } \kappa \longrightarrow +\infty\,.\end{aligned} \tag{4.1.20}$$

Theorem 4.1.4 will be useful in the proof of the next theorem which gives the asymptotic behavior of the order parameter ψ, when $(\psi, \mathbf{A})$ is a global minimizer.

Theorem 4.1.5. *Under the assumptions of Theorem 4.1.2, if $(\psi, \mathbf{A})$ is a minimizer of* (4.1.1) *and $\mathcal{D}$ is a regular set such that $\overline{\mathcal{D}} \subset \Omega$, then*

$$\int_{\mathcal{D}} |\psi(x)|^4\,dx = -\int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2 \left\{2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) - 1\right\}\,dx + o\,(1)\,, \quad \text{as } \kappa \longrightarrow +\infty\,. \tag{4.1.21}$$

Formula (4.1.21) indicates that ψ is asymptotically localized in the region where $a > 0$. When $a(x,\kappa) = 1$, Theorem 4.1.5 was proved in [5].

The techniques that we are going to use here are inspired from those of [5] and [6] (where the case $a = 1$ was treated). At a technical level, our proof is slightly different than the proofs in [5, 18, 13] since we do not use the uniform elliptic estimates. These important estimates are frequently used in the papers about the Ginzburg-Landau functional (see [14]) with a constant pinning term. They appeared first in [35] and were then extended to the full regime in [15].

Compared with other papers studying the pinned functional, one novelty here is that the pinning term has no definite sign, another one being the consideration of a variable (and a potentially vanishing) applied magnetic field.

The rest of this paper is devoted to the study of third critical field, i.e. the field above which the normal state $(0,\mathbf{F})$ is the only critical point of the functional in (4.1.1), in the case when the pining term a is independent of κ (i.e. $a(x,\kappa)=a(x)$). We define the set :

$$\mathcal{N}^{\mathrm{cp}}(\kappa)=\{H>0:\mathcal{E}_{\kappa,H,a,B_0}\text{ has a non-normal critical point}\}\,. \tag{4.1.22}$$

Notice that the above set is bounded (see Theorem 4.8.5). We also introduce the two sets :

$$\mathcal{N}(\kappa)=\{H>0:\mathcal{E}_{\kappa,H,a,B_0}\text{ has a non-normal minimizer}\}\,. \tag{4.1.23}$$

$$\mathcal{N}^{\mathrm{loc}}(\kappa)=\{H>0:\mu_1(\kappa,H)<0\}\,. \tag{4.1.24}$$

Here, $\mu_1(\kappa,H)$ is the ground state energy of the semi-bounded quadratic form

$$\mathcal{Q}^{\Omega}_{\kappa H\mathbf{F},-\kappa^2 a}(\phi)=\int_\Omega\left(|(\nabla-i\kappa H\mathbf{F})\phi|^2-\kappa^2\,a(x,\kappa)|\phi|^2\right)\,dx\,, \tag{4.1.25}$$

i.e.

$$\mu_1(\kappa,H)=\inf_{\substack{\phi\in H^1(\Omega)\\ \phi\neq 0}}\left(\frac{\mathcal{Q}^{\Omega}_{\kappa H\mathbf{F},-\kappa^2 a}(\phi)}{\|\phi\|^2_{L^2(\Omega)}}\right)\,. \tag{4.1.26}$$

Note that $\mu_1(\kappa,H)$ is the lowest eigenvalue of $P^{\Omega}_{\kappa H\mathbf{F},-\kappa^2 a}$. Here, we refer to [11, 32, 38, 42] for previous contributions.
We introduce the following critical fields (cf. e.g.[16, 35]) .

$$\overline{H}^{cp}_{C_3}(\kappa)=\sup\mathcal{N}^{cp}(\kappa)\,,\qquad \underline{H}^{cp}_{C_3}(\kappa)=\inf\left(\mathbb{R}_+\setminus\mathcal{N}^{cp}(\kappa)\right),\tag{4.1.27}$$

$$\overline{H}_{C_3}(\kappa)=\sup\mathcal{N}(\kappa)\,,\qquad \underline{H}_{C_3}(\kappa)=\inf\left(\mathbb{R}_+\setminus\mathcal{N}(\kappa)\right),\tag{4.1.28}$$

$$\overline{H}^{loc}_{C_3}(\kappa)=\sup\mathcal{N}^{loc}(\kappa)\,,\qquad \underline{H}^{loc}_{C_3}(\kappa)=\inf\left(\mathbb{R}_+\setminus\mathcal{N}^{loc}(\kappa)\right).\tag{4.1.29}$$

Below $\underline{H}_{C_3}$, normal states will loose their stability and above $\overline{H}_{C_3}$, the normal state is (up to a gauge transformation) the only critical point of the functional in (4.1.1).

Our aim is to determine the asymptotics of all the critical fields as $\kappa\longrightarrow+\infty$. This involves spectral quantities related to three models depending on Γ being empty or not.
Let us introduce

$$\Theta_0=\inf_{\xi\in\mathbb{R}}\mu(\xi)\,,$$

where μ is the lowest eigen value of the operator

$$\mathfrak{h}^{N,\xi}:=-\frac{d^2}{dt^2}+(t+\xi)^2\qquad\text{in }L^2(\mathbb{R}_+)\,,$$

subject to the Neumann boundary condition $u'(0)=0$.

Theorem 4.1.6. *Suppose that* $\Gamma=\{x\in\Omega:B_0(x)=0\}=\varnothing$ *and that* $a\in C^1(\overline{\Omega})$ *satisfies* $\{a>0\}\neq\varnothing$. *Then, as* $\kappa\longrightarrow+\infty$, *all the six critical fields satisfy an asymptotic expansion in* *the form :*

$$H_{C_3}(\kappa) = \max\left(\sup_{x\in\Omega} \frac{a(x)}{|B_0(x)|}, \sup_{x\in\partial\Omega} \frac{a(x)}{\Theta_0|B_0(x)|}\right)\kappa + \mathcal{O}(\kappa^{\frac{1}{2}})\,. \tag{4.1.30}$$

We introduce

$$\lambda_0 = \inf_{\tau\in\mathbb{R}} \lambda(\tau)\,, \tag{4.1.31}$$

where $\lambda(\tau)$ is the lowest eigenvalue of the selfadjoint realization of the differential operator

$$M(\tau) = -\frac{d^2}{dt^2} + \frac{1}{4}(t^2+2\tau)^2 \qquad \text{in } L^2(\mathbb{R})\,. \tag{4.1.32}$$

We consider, for any $\theta \in (0,\pi)$ the bottom of the spectrum $\lambda(\mathbb{R}^2_+,\theta)$ of the operator

$$P^{\mathbb{R}^2_+}_{\mathbf{A}_{\mathrm{app},\theta},0} \quad \text{with} \quad \mathbf{A}_{\mathrm{app},\theta} = -\left(\frac{x_2^2}{2}\cos\theta, \frac{x_1^2}{2}\sin\theta\right)\,. \tag{4.1.33}$$

Theorem 4.1.7. *Suppose that $\Gamma = \{x : B_0(x) = 0\} \neq \varnothing$, that* (4.1.4) *holds and that $a \in C^1(\overline{\Omega})$ satisfies $\{a > 0\} \neq \varnothing$. As $\kappa \longrightarrow +\infty$, the six critical fields in* (4.1.27)-(4.1.29) *satisfy the asymptotic expansion :*

$$H_{C_3}(\kappa) = \max\left(\sup_{x\in\Gamma\cap\overline{\Omega}} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|}, \sup_{x\in\Gamma\cap\partial\Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+,\theta(x))^{\frac{3}{2}}|\nabla B_0(x)|}\right)\kappa^2 + \mathcal{O}\left(\kappa^{\frac{11}{6}}\right)\,.$$

Here $\theta(x)$ denotes the angle between $\nabla B_0(x)$ and the inward normal vector $-\nu(x)$.

Organization of the paper

The rest of the paper is split into eleven sections. Section 4.2 analyzes the model problem with a constant magnetic field and a constant pinning term. Section 4.3 establishes an upper bound on the ground state energy. Section 4.4 contains useful estimates on minimizers. The estimates in Section 4.4 are used in Section 4.5 to establish a lower bound of the ground state energy and to finish the proof of Theorem 4.1.2, Corollary 4.1.3 and Theorem 4.1.4. In Section 4.6, we discuss the conclusion in Theorem 4.1.2 by providing various examples of pinning terms obeying Assumption 4.1.1. Section 7 is devoted to the proof of Theorem 4.1.5. Section 4.8 generalizes a theorem of Giorgi-Phillips concerning the breakdown of superconductivity under a large applied magnetic field. Sections 4.9 and 4.10 are devoted to the proof of Theorem 4.1.6. The proof of Theorem 4.1.7 is the purpose of Sections 4.11 and 4.12.

Notation.

Throughout the paper, we use the following notation :

- If $b_1(\kappa)$ and $b_2(\kappa)$ are two positive functions on $[\kappa_0,+\infty)$, we write $b_1(\kappa) \ll b_2(\kappa)$ if $b_1(\kappa)/b_2(\kappa) \to 0$ as $\kappa \to \infty$.
- If $b_1(\kappa)$ and $b_2(\kappa)$ are two functions with $b_2(\kappa) \neq 0$, we write $b_1(\kappa) \sim b_2(\kappa)$ if $b_1(\kappa)/b_2(\kappa) \to 1$ as $\kappa \to \infty$.

- If $b_1(\kappa)$ and $b_2(\kappa)$ are two positive functions, we write $b_1(\kappa) \approx b_2(\kappa)$ if there exist positive constants c_1, c_2 and κ_0 such that $c_1 b_2(\kappa) \leq b_1(\kappa) \leq c_2 b_2(\kappa)$ for all $\kappa \geq \kappa_0$.

- Let $a_+(\widetilde{x}_0,\kappa) = [a(\widetilde{x}_0,\kappa)]_+$ and $a_-(\widetilde{x}_0,\kappa) = [a(\widetilde{x}_0,\kappa)]_-$ where, for any $x \in \mathbb{R}$, $[x]_+ = \max(x,0)$ and $[x]_- = \max(-x,0)$.

- Given $R > 0$ and $x = (x_1,x_2) \in \mathbb{R}^2$, $Q_R(x) = (-R/2+x_1, R/2+x_1)\times(-R/2+x_2, R/2+x_2)$ denotes the square of side length R centered at $x = (x_1,x_2)$ and we write $Q_R = Q_R(0)$.

4.2 A reference problem

The reference problem is obtained by freezing the pinning term and the magnetic field. This approximation will appear to be reasonable in squares avoiding the boundary and the zero set Γ of the magnetic field B_0.

4.2.1 A useful function

Consider $R > 0$, $b > 0$, $\zeta \in \{-1,+1\}$ and $\alpha \in \mathbb{R}$. We define the following Ginzburg-Landau energy with constant magnetic field on $H^1(Q_R)$ by

$$u \mapsto F^{\zeta,\alpha}_{b,Q_R}(u) = \int_{Q_R} \left(b|(\nabla - i\zeta\mathbf{A}_0)u|^2 + \frac{1}{2}\left(\alpha - |u|^2\right)^2 \right) dx\,, \tag{4.2.1}$$

where

$$\mathbf{A}_0(x) = \frac{1}{2}(-x_2, x_1)\,, \qquad \forall\, x = (x_1,x_2) \in \mathbb{R}^2\,. \tag{4.2.2}$$

We have two cases according to the sign of α :

Case 1. $\alpha > 0$:

We notice that

$$F^{\zeta,\alpha}_{b,Q_R}(u) = \alpha^2 F^{\zeta,1}_{\widetilde{b},Q_R}(\widetilde{u})\,, \tag{4.2.3}$$

where

$$\widetilde{b} = \frac{b}{\alpha} \qquad \text{and} \qquad \widetilde{u} = \frac{u}{\sqrt{\alpha}}\,. \tag{4.2.4}$$

We introduce the two ground state energies

$$e_N(b,R,\alpha) = \inf\left\{ F^{+1,\alpha}_{b,Q_R}(u) : u \in H^1(Q_R;\mathbb{C}) \right\} \tag{4.2.5}$$

$$e_D(b,R,\alpha) = \inf\left\{ F^{+1,\alpha}_{b,Q_R}(u) : u \in H^1_0(Q_R;\mathbb{C}) \right\}\,. \tag{4.2.6}$$

As $F^{+1,\alpha}_{b,Q_R}(u) = F^{-1,\alpha}_{b,Q_R}(\overline{u})$, it is immediate that,

$$\inf F^{+1,\alpha}_{b,Q_R}(u) = \inf F^{-1,\alpha}_{b,Q_R}(u)\,. \tag{4.2.7}$$

Using (4.2.5) and (4.2.6), we get from (4.2.3)

$$e_N(b,R,\alpha) = \alpha^2\, e_N\left(\frac{b}{\alpha}, R, 1\right) = \alpha^2\, e_N\left(\frac{b}{\alpha}, R\right)\,, \tag{4.2.8}$$

and

$$e_D(b,R,\alpha) = \alpha^2\, e_D\left(\frac{b}{\alpha}, R, 1\right) = \alpha^2\, e_D\left(\frac{b}{\alpha}, R\right)\,. \tag{4.2.9}$$

As a consequence of (4.2.3) and (4.2.4), $\widetilde{u}$ is a minimizer of $F^{\zeta,1}_{\widetilde{b},Q_R}$ if and only if u is a minimizer of $F^{\zeta,\alpha}_{b,Q_R}$. In particular any minimizer of $F^{\zeta,\alpha}_{b,Q_R}$ satisfies

$$|u| \leq \sqrt{\alpha}\,. \tag{4.2.10}$$

Recall from [18, Theorem 2.1] that,

$$\hat{f}\,(\mathfrak{b}) = \lim_{R\longrightarrow\infty} \frac{e_D(\mathfrak{b},R)}{R^2}\,. \tag{4.2.11}$$

The next proposition was proved in [6, Lemma 2.2, Proposition 2.4] in the case $\alpha = 1$. It's present form can be deduced immediately from (4.2.8).

Proposition 4.2.1. *For all $M > 0$, there exist universal constants C_M and R_M such that $\forall R \geq R_M$, $\forall\, b > 0$, $\forall\, \alpha > 0$ such that $0 < \dfrac{b}{\alpha} \leq M$, we have*

$$e_N(b,R,\alpha) \geq\; e_D\,(b,R,\alpha) - C_M\alpha^2 R\left(\frac{b}{\alpha}\right)^{\frac{1}{2}} \tag{4.2.12}$$

$$\alpha^2\hat{f}\left(\frac{b}{\alpha}\right) \leq \frac{e_D(b,R,\alpha)}{R^2} \leq \alpha^2\hat{f}\left(\frac{b}{\alpha}\right) + C_M\frac{\alpha^{\frac{3}{2}}\sqrt{b}}{R}. \tag{4.2.13}$$

Case 2. $\alpha \leq 0$:

When $\alpha \leq 0$, we write $\alpha = -\alpha_0$, $\alpha_0 \geq 0$ and (4.2.1) becomes

$$F^{\zeta,\alpha}_{b,Q_R}(u) = \int_{Q_R}\left(b|(\nabla - i\zeta\mathbf{A}_0)u|^2 + \frac{1}{2}\left(\alpha_0 + |u|^2\right)^2\right)\,dx\,. \tag{4.2.14}$$

It is clear that,

$$F^{\zeta,\alpha}_{b,Q_R}(u) \geq \frac{1}{2}\alpha_0^2 R^2 \qquad \text{and} \qquad F^{\zeta,\alpha}_{b,Q_R}(0) = \frac{1}{2}\alpha_0^2 R^2\,.$$

As a consequence, we have

$$\frac{1}{2}\alpha_0^2 R^2 \leq e_D(b,R,\alpha) \leq F^{\zeta,\alpha}_{b,Q_R}(0) = \frac{1}{2}\alpha_0^2 R^2\,.$$

When $\alpha = 0$, it is easy to show that

$$F^{\zeta,\alpha}_{b,Q_R}(u) = 0\,.$$

Notice that the only minimizer of $F^{\zeta,\alpha}_{b,Q_R}$ is $u = 0$. Thus, for any $\alpha \leq 0$, we obtain

$$\frac{e_D(b,R,\alpha)}{R^2} = \frac{1}{2}\alpha^2\,. \tag{4.2.15}$$

4.3 Upper bound of the energy

The aim of this section is to give an upper bound of the ground state energy $\mathrm{E_g}(\kappa, H, a, B_0)$ introduced in (4.1.14) under Assumption (4.1.15). For this we cover Ω by (the closure of) disjoint open squares $(Q_\ell(\gamma))_\gamma$ whose centers γ belong to a square lattice $\Gamma_\ell = \ell\mathbb{Z} \times \ell\mathbb{Z}$.

We will get an upper bound by matching together approximate minimizers, in each square $Q_\ell(\gamma)$ contained in Ω, obtained by freezing the pinning term and the magnetic field at a suitable point $\tilde{\gamma}$. The size ℓ of the square will be chosen as a function of κ. We start with estimates in a given square $Q_\ell(x_0)$ and will take later $x_0 = \gamma$.

About Assumption (A_4)**.**
We first explain what was meant in Assumption (A_4). By $\mathcal{L}(\partial\{a > 0\}) \leq C_1\kappa^{\frac{1}{2}}$ we mean the existence of $C_2 > 0$ and κ_0 such that :

$$\forall \kappa \geq \kappa_0\,,\ \forall \ell \leq C_2\kappa^{-\frac{1}{2}}\,,\ \mathrm{card}\,\{\gamma \in \Gamma_\ell \cap \Omega \text{ with } Q_\ell(\gamma) \cap \partial\{a > 0\} \cap \Omega \neq \emptyset\} \leq C_1\kappa^{\frac{1}{2}}\ell^{-1}\,. \tag{4.3.1}$$

Using Assumption (4.1.9), for any $\widetilde{x}_0 \in \overline{Q_\ell(x_0)}$ and $\kappa \geq \kappa_0$, we observe that,

$$|a(x,\kappa) - a(\widetilde{x}_0,\kappa)| \leq \left(\sup_x |\nabla_x\, a(x,\kappa)|\right)\, |x - x_0| \leq \frac{\ell}{\sqrt{2}}\, L(\kappa)\,, \qquad \forall x \in Q_\ell(x_0)\,. \tag{4.3.2}$$

Definition 4.3.1 (ρ-admissible)**.** *Let $\rho \in (0,1)$. We say that triple $(\ell, x_0, \widetilde{x}_0)$ is ρ-admissible if $\overline{Q_\ell(x_0)} \subset \{|B_0| > \rho\} \cap \Omega$ and $\widetilde{x}_0 \in \overline{Q_\ell(x_0)}$. In this case, we also say that the pair (ℓ, x_0) is ρ-admissible and the corresponding square $Q_\ell(x_0)$ is ρ admissible.*

We recall from [6, Section 3] the definition of the test function,

$$\widetilde{w}_{\ell,x_0,\widetilde{x}_0}(x) = \begin{cases} e^{i\kappa H\varphi_{x_0,\widetilde{x}_0}}\widetilde{u}_R\left(\frac{R}{\ell}(x - x_0)\right) & \text{if } x \in Q_\ell(x_0) \subset \{B_0 > \rho\} \cap \Omega \\ e^{i\kappa H\varphi_{x_0,\widetilde{x}_0}}\overline{\widetilde{u}}_R\left(\frac{R}{\ell}(x - x_0)\right) & \text{if } x \in Q_\ell(x_0) \subset \{B_0 < -\rho\} \cap \Omega\,, \end{cases} \tag{4.3.3}$$

where $\widetilde{u}_R \in H^1_0(\Omega)$ is a minimizer of $F^{+1,1}_{b,Q_R}$ satisfying by (4.2.10) $|\widetilde{u}_R| \leq 1$ and $\varphi_{x_0,\widetilde{x}_0}$ is the function introduced in [5, Lemma A.3] that satisfies

$$|\mathbf{F}(x) - B_0(\widetilde{x}_0)\mathbf{A}_0(x - x_0) - \nabla\varphi_{x_0,\widetilde{x}_0}(x)| \leq C\,\ell^2, \qquad \forall x \in Q_\ell(x_0)\,. \tag{4.3.4}$$

Here $B_0 = \mathrm{curl}\,\mathbf{F}$ and $\mathbf{A}_0$ is the magnetic potential introduced in (4.2.2).

Let us introduce the function :

$$w_{\ell,x_0,\widetilde{x}_0}(x)=\sqrt{a_+(\widetilde{x}_0,\kappa)}\,\widetilde{w}_{\ell,x_0,\widetilde{x}_0}(x)\,,\qquad \forall x\in Q_\ell(\widetilde{x}_0)\,. \tag{4.3.5}$$

Using the bound $|\widetilde{w}_{\ell,x_0,\widetilde{x}_0}|\leq 1$, which is immediately deduced from the bound of $|\widetilde{u}_R|$, we get from (4.3.5),

$$|w_{\ell,x_0,\widetilde{x}_0}|^2\leq a_+(\widetilde{x}_0,\kappa)\,. \tag{4.3.6}$$

Proposition 4.3.2. *Under Assumptions* (4.1.4)-(4.1.7)*, there exist positive constants C and κ_0 such that if $\kappa\geq\kappa_0$, $\ell\in(0,1)$, $\delta\in(0,1)$, $\rho>0$, $\ell^2\kappa H\rho>1$ and $(\ell,x_0,\widetilde{x}_0)$ is a ρ-admissible triple, then,*

$$\begin{aligned}\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0},\mathbf{F};a,Q_\ell(x_0))\leq(1+\delta)\kappa^2\left[a_+(\widetilde{x}_0,\kappa)^2\hat{f}\left(\frac{H\,|B_0(\widetilde{x}_0)|}{\kappa\,a_+(\widetilde{x}_0,\kappa)}\right)+\frac{1}{2}a_-(\widetilde{x}_0,\kappa)^2\right]\\+C\left(\frac{1}{\kappa\ell}+\delta^{-1}\ell^2L(\kappa)^2+\delta^{-1}\kappa^2\ell^4\right)\kappa^2\,.\end{aligned} \tag{4.3.7}$$

Proof.
Let

$$R=\ell\sqrt{\kappa H|B_0(\widetilde{x}_0)|}\qquad\text{and}\qquad b=\frac{H\,|B_0(\widetilde{x}_0)|}{\kappa}\,. \tag{4.3.8}$$

First we estimate $\frac{\kappa^2}{2}\int_{Q_\ell(x_0)}(a(x,\kappa)-|w_{\ell,x_0,\widetilde{x}_0}|^2)^2\,dx$ from above. Using (4.3.2), we get the existence of a constant $C>0$ such that for any $\delta\in(0,1)$ and any $\kappa\geq\kappa_0$,

$$\begin{aligned}\frac{\kappa^2}{2}\int_{Q_\ell(x_0)}\left(a(x,\kappa)-|w_{\ell,x_0,\widetilde{x}_0}|^2\right)^2\,dx&\leq(1+\delta)\frac{\kappa^2}{2}\int_{Q_\ell(x_0)}\left(a(\widetilde{x}_0,\kappa)-|w_{\ell,x_0,\widetilde{x}_0}|^2\right)^2\,dx\\&\quad+(1+\delta^{-1})\frac{\kappa^2}{2}\int_{Q_\ell(x_0)}\left(a(\widetilde{x}_0,\kappa)-a(x,\kappa)\right)^2\,dx\\&\leq(1+\delta)\frac{\kappa^2}{2}\int_{Q_\ell(x_0)}\left(a(\widetilde{x}_0,\kappa)-|w_{\ell,x_0,\widetilde{x}_0}|^2\right)^2\,dx\\&\quad+C\delta^{-1}\kappa^2\ell^4L(\kappa)^2\,.\end{aligned} \tag{4.3.9}$$

The estimate of $\int_{Q_\ell(x_0)}|(\nabla-i\kappa H\mathbf{F})w_{\ell,x_0,\widetilde{x}_0}|^2\,dx$ from above is the same as in [6, Proposition 3.1]. We have

$$\begin{aligned}&\int_{Q_\ell(x_0)}|(\nabla-i\kappa H\mathbf{F})w_{\ell,x_0,\widetilde{x}_0}|^2\,dx\\&\qquad\leq(1+\delta)\int_{Q_\ell(x_0)}\left|\left(\nabla-i\kappa H(B_0(\widetilde{x}_0)\mathbf{A}_0(x-x_0)+\nabla\varphi_{x_0,\widetilde{x}_0})\right)w_{\ell,x_0,\widetilde{x}_0}\right|^2\,dx\\&\qquad\quad+C\delta^{-1}\kappa^4\ell^6|\,w_{\ell,x_0,\widetilde{x}_0}|^2\,.\end{aligned} \tag{4.3.10}$$

From (4.1.10), by collecting (4.3.9), (4.3.10) and (4.3.6), we find that,

$$\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0}, \mathbf{F}; a, Q_\ell(x_0)) \leq (1+\delta)\mathcal{E}_0\big(w_{\ell,x_0,\widetilde{x}_0},\, B_0(\widetilde{x}_0)\mathbf{A}_0(x-x_0) + \nabla\varphi_{x_0,\widetilde{x}_0}; a(\widetilde{x}_0,\kappa), Q_\ell(x_0)\big) + C\delta^{-1}(\kappa^2\ell^4 L(\kappa)^2 + \kappa^4\ell^6\, a_+(\widetilde{x}_0,\kappa))\,. \quad (4.3.11)$$

As we did in [6], we use the change of variable $y = \frac{R}{\ell}(x - x_0)$ and obtain

$$\mathcal{E}_0\big(w_{\ell,x_0,\widetilde{x}_0},\, B_0(\widetilde{x}_0)\mathbf{A}_0(x-x_0) + \nabla\varphi_{x_0,\widetilde{x}_0}; a(\widetilde{x}_0,\kappa), Q_\ell(x_0)\big)$$
$$= \int_{Q_R}\left[a_+(\widetilde{x}_0,\kappa)\left|\left(\frac{R}{\ell}\nabla - i\frac{R}{\ell}\zeta_\ell\,\mathbf{A}_0(y)\right)\widetilde{u}_R(y)\right|^2 + \frac{\kappa^2}{2}\left(a(\widetilde{x}_0,\kappa) - a_+(\widetilde{x}_0,\kappa)\,|\widetilde{u}_R(y)|^2\right)^2\right]\frac{\ell^2}{R^2}dy.$$

Here, we denote by ζ_ℓ the sign of $B_0(x_0)$.
We distinguish between two cases :
Case 1 : When $a(\widetilde{x}_0,\kappa) > 0$, we get

$$\mathcal{E}_0\big(w_{\ell,x_0,\widetilde{x}_0}, B_0(\widetilde{x}_0)\mathbf{A}_0(x-x_0) + \nabla\varphi_{x_0,\widetilde{x}_0}; a(\widetilde{x}_0,\kappa), Q_\ell(x_0)\big) = \frac{a(\widetilde{x}_0,\kappa)^2}{b}F^{\zeta_\ell,1}_{b/a(\widetilde{x}_0,\kappa),Q_R}(\widetilde{u}_R)\,.$$

From (4.2.7) and (4.2.8), we obtain,

$$\mathcal{E}_0\big(w_{\ell,x_0,\widetilde{x}_0}, B_0(\widetilde{x}_0)\mathbf{A}_0(x-x_0) + \nabla\varphi_{x_0,\widetilde{x}_0}; a(\widetilde{x}_0,\kappa), Q_\ell(x_0)\big) = \frac{1}{b}e_D(b,R,a(\widetilde{x}_0,\kappa))\,. \quad (4.3.12)$$

As a consequence of the upper bound in (4.2.13), the ground state energy $e_D(b,R,a(\widetilde{x}_0,\kappa))$ in (4.3.12) is bounded for all $b > 0$ and $R \geq 1$ by :

$$e_D(b,R,a(\widetilde{x}_0,\kappa)) \leq a(\widetilde{x}_0,\kappa)^2\,R^2\,\hat{f}\left(\frac{b}{a(\widetilde{x}_0,\kappa)}\right) + C_M a(\widetilde{x}_0,\kappa)^{\frac{3}{2}}\,R\,\sqrt{b}\,. \quad (4.3.13)$$

With the choice of R in (4.3.8), we have effectively $R \geq 1$ which follows from the assumption $R \geq \ell\sqrt{\kappa H\rho} > 1$.
We get from (4.3.12) and (4.3.13) the estimate

$$\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0}, \zeta_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0) + \nabla\varphi_{x_0,\widetilde{x}_0}; a(\widetilde{x}_0,\kappa), Q_\ell(x_0)) \leq a(\widetilde{x}_0,\kappa)^2\frac{R^2}{b}\hat{f}\left(\frac{b}{a(\widetilde{x}_0,\kappa)}\right) + C_M\frac{a(\widetilde{x}_0,\kappa)^{\frac{3}{2}}\,R}{\sqrt{b}}\,, \quad (4.3.14)$$

with (b,R) defined in (4.3.8).
By collecting the estimates in (4.3.11)-(4.3.14) we get,

$$\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0}, \mathbf{F}; a(\widetilde{x}_0,\kappa), Q_\ell(x_0)) \leq (1+\delta)\,a(\widetilde{x}_0,\kappa)^2\,\frac{R^2}{b}\,\hat{f}\left(\frac{b}{a(\widetilde{x}_0,\kappa)}\right) + C_M\frac{\overline{a}^{\frac{3}{2}}\,R}{\sqrt{b}} + C\delta^{-1}(\kappa^2\ell^4 L(\kappa)^2 + \kappa^4\ell^6\overline{a})\,. \quad (4.3.15)$$

Here, we have used the fact that $a(\widetilde{x}_0,\kappa)\leq \sup\limits_{x\in\overline{\Omega},\,\kappa\geq\kappa_0} a(x,\kappa)=\overline{a}$.

Case 2 : When $a(\widetilde{x}_0,\kappa)\leq 0$, we have,

$$\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0},\mathbf{F};a(\widetilde{x}_0,\kappa),Q_\ell(x_0))=\frac{\kappa^2}{2}\int_{Q_\ell(x_0)}a(x,\kappa)^2\,dx\,.$$

From (4.3.2), we get the existence of a constant $C>0$ such that for any $\delta\in(0,1)$,

$$\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0},\mathbf{F};a(\widetilde{x}_0,\kappa),Q_\ell(x_0))\leq(1+\delta)\,\frac{\kappa^2}{2}\,a(\widetilde{x}_0,\kappa)^2\ell^2+C\,\delta^{-1}\,\kappa^2\ell^4L(\kappa)^2\,. \tag{4.3.16}$$

The results of cases 1-2, we obtain,

$$\begin{aligned}\mathcal{E}_0(w_{\ell,x_0,\widetilde{x}_0},\mathbf{F};a(\widetilde{x}_0,\kappa),Q_\ell(x_0))\leq(1+\delta)\kappa^2\left[a_+(\widetilde{x}_0,\kappa)^2\hat{f}\left(\frac{H\,|B_0(\widetilde{x}_0)|}{\kappa\,a_+(\widetilde{x}_0,\kappa)}\right)+\frac{1}{2}a_-(\widetilde{x}_0,\kappa)^2\right]\ell^2\\+C\left(\frac{\kappa}{\ell}\,\overline{a}^{\frac{3}{2}}+\delta^{-1}\kappa^2\ell^2L(\kappa)^2+\delta^{-1}\kappa^4\ell^4\,\overline{a}\right)\ell^2\,,\end{aligned} \tag{4.3.17}$$

which finishes the proof of Proposition 4.3.2. $\square$

Application 4.3.3.
We select ℓ, ρ, δ and the constraint on $L(\kappa)$ as follows :

$$\ell=\kappa^{-\frac{7}{12}}\,,\qquad \rho=\kappa^{-\frac{17}{24}}\,,\qquad L(\kappa)\leq C\,\kappa^{\frac{1}{2}}\,. \tag{4.3.18}$$

and

$$\delta=\kappa^{-\frac{1}{12}} \tag{4.3.19}$$

Under Assumption (4.1.15)*, this choice permits to verify the assumptions in Proposition 4.3.2 and to obtain error terms of order $o(\kappa^2)$. We have indeed as $\kappa\longrightarrow\infty$*

$$\frac{\kappa}{\ell}=\kappa^{\frac{19}{12}}\ll\kappa^2\,,$$

$$\delta^{-1}\kappa^2\ell^2\,L(\kappa)^2\leq\kappa^{\frac{23}{12}}\ll\kappa^2\,,$$

$$\delta^{-1}\kappa^4\ell^4=\kappa^{\frac{21}{12}}\ll\kappa^2\,,$$

$$\ell^2\kappa H\rho=\kappa^{\frac{3}{24}}\gg 1\,.$$

Theorem 4.3.4. *Under Assumptions* (4.1.4)-(4.1.8)*, if* (4.1.15) *holds and $L(\kappa)\leq C\,\kappa^{\frac{1}{2}}$, then, the ground state energy $\mathrm{E}_{\mathrm{g}}(\kappa,H,a,B_0)$ in* (4.1.14) *satisfies*

$$\begin{aligned}\mathrm{E}_{\mathrm{g}}(\kappa,H,a,B_0)\leq\kappa^2\int_{\{a(x,\kappa)>0\}}a(x,\kappa)^2\,\hat{f}\left(\frac{H\,|B_0(x)|}{\kappa\,a(x,\kappa)}\right)\,dx\\+\frac{\kappa^2}{2}\int_{\{a(x,\kappa)\leq 0\}}a(x,\kappa)^2\,dx+o(\kappa^2)\,,\quad\text{as }\kappa\longrightarrow\infty\,.\end{aligned} \tag{4.3.20}$$

Proof. Let $\ell\in(0,1)$, δ and ρ be chosen as in (4.3.18) and (4.3.19). We consider the lattice

$\Gamma_\ell := \ell\mathbb{Z} \times \ell\mathbb{Z}$ and write, for $\gamma \in \Gamma_\ell$, $Q_{\gamma,\ell} = Q_\ell(\gamma)$. In the next decomposition we keep the ρ-admissible boxes $Q_\ell(\gamma)$ in Ω which in addition are either contained in $\{a > 0\}$ or in $\{a \leq 0\}$. Hence we introduce

$$\mathcal{I}^+_{\ell,\rho} = \left\{\gamma;\, \overline{Q_{\gamma,\ell}} \subset \Omega \cap \{|B_0| > \rho\,; a > 0\}\right\}, \quad \mathcal{I}^-_{\ell,\rho} = \left\{\gamma;\, \overline{Q_{\gamma,\ell}} \subset \Omega \cap \{|B_0| > \rho\,; a \leq 0\}\right\}, \tag{4.3.21}$$

and

$$N^+ = \text{card}\, \mathcal{I}^+_{\ell,\rho}\,, \qquad N^- = \text{card}\, \mathcal{I}^-_{\ell,\rho}\,. \tag{4.3.22}$$

Under Assumption (4.1.8), we have,

$$N^+ + N^- = |\Omega|\ell^{-2} + \mathcal{O}(\kappa^{\frac{1}{2}}\ell^{-1} + \ell^{-1} + \rho\ell^{-2})\,, \qquad \text{as } \kappa \to +\infty\,. \tag{4.3.23}$$

In (4.3.23), $\kappa^{\frac{1}{2}}\ell^{-1}$ appears when treating the boundary of the set $\{a(x,\kappa) > 0\}$ (using Assumption (A_4) as explained in (4.3.1)), ℓ^{-1} appears in the treatment of the boundary and $\rho\ell^{-2}$ appears when treating the neighborhood of Γ.
In each ρ-admissible $Q_\ell(\gamma)$, we consider some $\widetilde{\gamma}$ (to be chosen later) such that $(\ell,\gamma,\widetilde{\gamma})$ be a ρ-admissible triple. We consider $w_{\ell,\gamma,\widetilde{\gamma}}$ and extend it by 0 outside of $Q_{\gamma,\ell}$, keeping the same notation for this extension. Then we define

$$s(x) = \sum_{\gamma \in \mathcal{I}^+_{\ell,\rho} \cup \mathcal{I}^-_{\ell,\rho}} w_{\ell,\gamma,\widetilde{\gamma}}(x)\,. \tag{4.3.24}$$

We compute the Ginzburg-Landau energy of the test configuration $(s, \mathbf{F})$ in Ω. Since $\operatorname{curl}\mathbf{F} = B_0$, we get,

$$\mathcal{E}_{\kappa,H,a,B_0}(s,\mathbf{F},\Omega) = \sum_{\gamma \in \mathcal{I}^+_{\ell,\rho} \cup \mathcal{I}^-_{\ell,\rho}} \mathcal{E}_0(w_{\ell,\gamma,\widetilde{\gamma}}, \mathbf{F}; a(\widetilde{\gamma},\kappa), Q_{\gamma,\ell})\,. \tag{4.3.25}$$

Notice that for any $\widetilde{\gamma} \in Q_{\gamma,\ell}$, $a(\widetilde{\gamma},\kappa)$ satisfies (4.3.2) with $x = \gamma$ and $\widetilde{x}_0 = \widetilde{\gamma}$, and $B_0(\widetilde{\gamma})$ satisfies (4.3.4). We recall that $\hat{f}$ is a continuous, non-decreasing function (see [6, Theorem 2.1]) and that B_0 and $a(\cdot,\kappa)$ are in C^1. Then, in each box $Q_{\gamma,\ell}$, we select $\widetilde{\gamma} \in \overline{Q_{\gamma,\ell}}$ such that

$$|a(\widetilde{\gamma},\kappa)|^2\, \hat{f}\left(\frac{H\,B_0(\widetilde{\gamma})}{\kappa\, a(\widetilde{\gamma},\kappa)}\right) = \inf_{\widehat{\gamma} \in Q_{\gamma,\ell}} |a(\widehat{\gamma},\kappa)|^2\, \hat{f}\left(\frac{H\,B_0(\widehat{\gamma})}{\kappa\, a(\widehat{\gamma},\kappa)}\right) \quad (\text{if } \gamma \in \mathcal{I}^+_{\ell,\rho})$$

and

$$|a(\widetilde{\gamma},\kappa)|^2 = \inf_{\widehat{\gamma} \in Q_{\gamma,\ell}} |a(\widehat{\gamma},\kappa)|^2 \quad (\text{if } \gamma \in \mathcal{I}^-_{\ell,\rho})\,.$$

Using Proposition 4.3.2 and noticing that $|Q_{\gamma,\ell}| = \ell^2$, we get the existence of $C > 0$ such that, for any $\delta \in (0,1)$

$$\sum_{\gamma\in\mathcal{I}^+_{\ell,\rho}\cup\mathcal{I}^-_{\ell,\rho}} \mathcal{E}_0(w_{\ell,\gamma,\widetilde{\gamma}},\mathbf{F};a(\widetilde{\gamma},\kappa),Q_{\gamma,\ell}) \leq \kappa^2(1+\delta)\sum_{\gamma\in\mathcal{I}^+_{\ell,\rho}} \inf_{\widehat{\gamma}\in Q_{\gamma,\ell}} [a(\widehat{\gamma},\kappa)]_+^2\,\hat{f}\left(\frac{H\,B_0(\widehat{\gamma})}{\kappa\,a(\widehat{\gamma},\kappa)}\right)\ell^2$$
$$+\kappa^2(1+\delta)\sum_{\gamma\in\mathcal{I}^-_{\ell,\rho}} \inf_{\widehat{\gamma}\in Q_{\gamma,\ell}} \frac{[a(\widehat{\gamma},\kappa)]_-^2}{2}\ell^2 + C\sum_{\gamma\in\mathcal{I}^+_{\ell,\rho}\cup\mathcal{I}^-_{\ell,\rho}} \left(\frac{\kappa}{\ell}+\delta^{-1}\kappa^2\ell^2 L(\kappa)^2+\delta^{-1}\kappa^4\ell^4\right)\ell^2\,. \tag{4.3.26}$$

We recognize the lower Riemann sum of the function $x\longmapsto [a(x,\kappa)]_+^2\,\hat{f}\left(\frac{H\,B_0(x)}{\kappa\,a(x,\kappa)}\right)$ in $(\cup_{\gamma\in\mathcal{I}^+_{\ell,\rho}}Q_{\gamma,\ell})$ and the function $x\longmapsto [a(x,\kappa)]_-^2$ in $(\cup_{\gamma\in\mathcal{I}^-_{\ell,\rho}}Q_{\gamma,\ell})$. Notice that $\{\cup_{\gamma\in\mathcal{I}_{\ell,\rho}}Q_{\gamma,\ell}\}\subset\Omega$. Thanks to Application 4.3.3, using (4.3.23) and the non negativity of $\hat{f}$, we get by collecting (4.3.25)-(4.3.26) that,

$$\mathcal{E}_{\kappa,H,a,B_0}(s,\mathbf{F},\Omega) \leq \kappa^2\int_{\{a(x,\kappa)>0\}} a(x,\kappa)^2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)dx + \frac{\kappa^2}{2}\int_{\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\,dx + C\,\kappa^{\frac{23}{12}}\,. \tag{4.3.27}$$

Since $(\psi,\mathbf{A})$ is a minimizer of the functional $\mathcal{E}_{\kappa,H,a,B_0}$ in (4.1.1), we get

$$\mathrm{E_g}(\kappa,H,a,B_0) \leq \mathcal{E}_{\kappa,H,a,B_0}(s,\mathbf{F},\Omega)\,.$$

This finishes the proof of Theorem 4.3.4. □

4.4 A priori estimates of minimizers

The aim of this section is to give a priori estimates for the solutions of the Ginzburg-Landau equations (4.1.12). In the case when $a(x,\kappa)=1$ the starting point is an L^∞ estimate of ψ. This estimate can be easly extended in the general case considered in this paper when $(4.1.12)_a$ and $(4.1.12)_c$ hold. Let us introduce :

$$\overline{a}(\kappa)=\sup_{x\in\overline{\Omega}} a(x,\kappa)\,. \tag{4.4.1}$$

Proposition 4.4.1. *Let $\kappa>0$; if $(\psi,\mathbf{A})$ is a critical point (see (4.1.12)), then,*

$$|\psi(x)|^2 \leq \max\{\overline{a}(\kappa),0\}\,,\qquad \forall x\in\overline{\Omega}\,. \tag{4.4.2}$$

Proof. We distinguish between two cases :
Case 1 : $\overline{a}(\kappa)\leq 0$ **.**
Multiplying the equation for ψ in $(4.1.12)_a$ by $\overline{\psi}$ and integrating over Ω, we get

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A})\psi|^2\,dx = \kappa^2\int_\Omega (a(x,\kappa)-|\psi|^2)|\psi|^2\,dx\,. \tag{4.4.3}$$

Since $(a(x,\kappa)-|\psi|^2)\leq -|\psi|^2$, we obtain that $|\psi|^2=0$ almost everywhere.
Case 2 : $\overline{a}(\kappa)>0$.

We will show that $\psi \in C^0(\overline{\Omega})$. In fact, $(\psi, \mathbf{A})$ satisfies $(4.1.12)_a$, $\psi \in L^p(\Omega)$ for all $2 \leq p < +\infty$ and $\mathbf{A} \in H^1_{\rm div}(\Omega) \hookrightarrow L^p(\Omega)$. Thus, $\psi \in W^{2,q}(\Omega)$ for all $q < 2$. As a consequence of the continuous Sobolev embedding of $W^{j+m,q}(\Omega)$ into $C^j(\overline{\Omega})$ for any $q > \frac{2}{m}$, we obtain that $\psi \in C^0(\overline{\Omega})$. Define for any $\kappa > 0$ the following open set :

$$\Omega_+ = \left\{ x \in \Omega : \, |\psi(x)| > \sqrt{\overline{a}(\kappa)} \right\} \,, \tag{4.4.4}$$

and the following functions on Ω_+

$$\phi = \frac{\psi}{|\psi|}\,, \qquad \widehat{\psi} = \left[|\psi| - \sqrt{\overline{a}(\kappa)} \right]_+ \phi\,.$$

It is clear that

$$\nabla \left[|\psi| - \sqrt{\overline{a}(\kappa)} \right]_+ = 1_{\Omega_+} \nabla \left(|\psi| - \sqrt{\overline{a}(\kappa)} \right) = 1_{\Omega_+} \nabla |\psi|\,.$$

Notice that $\psi \in H^1(\Omega)$, so applying [14, Proposition 3.1.2], we get the property that $\nabla \left[|\psi| - \sqrt{\overline{a}(\kappa)} \right]_+ \in L^2(\Omega)$, which implies that $\left[|\psi| - \sqrt{\overline{a}(\kappa)} \right]_+ \in H^1(\Omega)$.
We introduce an increasing cut-off function $\chi \in C^\infty(\mathbb{R})$ such that,

$$\chi(t) = \begin{cases} 0 & \text{for } \ t \leq \frac{1}{4}\sqrt{\overline{a}(\kappa)} \\ 1 & \text{for } t \geq \frac{3}{4}\sqrt{\overline{a}(\kappa)}\,, \end{cases} \tag{4.4.5}$$

and define

$$\widehat{\phi} = \chi(|\psi|) \frac{\psi}{|\psi|}\,. \tag{4.4.6}$$

Since $\chi(|\psi|)\frac{\psi}{|\psi|}$ is smooth with bounded derivatives and $\psi \in H^1(\Omega)$, the chain rule gives that $\widehat{\phi} \in H^1(\Omega)$. Furthermore,

$$(\nabla - i\kappa H \mathbf{A})\widehat{\psi} = 1_{\Omega_+} \widehat{\phi}\, \nabla |\psi| + \left[|\psi| - \sqrt{\overline{a}(\kappa)} \right]_+ (\nabla - i\kappa H\mathbf{A})\widehat{\phi}. \tag{4.4.7}$$

Using (4.4.5) and (4.4.6), we get

$$1_{\Omega_+}(\nabla - i\kappa H\mathbf{A})\psi = 1_{\Omega_+}(\nabla - i\kappa H\mathbf{A})(|\psi|\widehat{\phi}) = 1_{\Omega_+}\{\widehat{\phi}\,\nabla|\psi| + |\psi|(\nabla - i\kappa H\mathbf{A})\widehat{\phi}\}\,. \tag{4.4.8}$$

We have on Ω_+ that $|\phi| = |\widehat{\phi}| = 1$. Therefore

$$\begin{aligned} \phi \nabla \overline{\phi} + \overline{\phi \nabla \overline{\phi}} &= \phi \nabla \overline{\phi} + \overline{\phi} \nabla \phi \\ &= \nabla |\phi|^2 \\ &= 0\,. \end{aligned}$$

So, $\mathrm{Re}(1_{\Omega_+} \phi \nabla \overline{\phi}) = 0$. This implies by using (4.4.7) and (4.4.8) that

$$\mathrm{Re}\left\{ \overline{(\nabla - i\kappa H\mathbf{A})\widehat{\psi}} \cdot (\nabla - i\kappa H\mathbf{A})\psi \right\} = 1_{\Omega_+} \left(|\nabla|\psi||^2 + \left(|\psi| - \sqrt{\overline{a}(\kappa)} \right) |\psi| |(\nabla - i\kappa H\mathbf{A})\widehat{\phi}|^2 \right).$$

Multiplying $(4.1.12)_a$ by $\overline{\widehat{\psi}}$ and using $(4.1.12)_c$, it results from an integration by parts over Ω that

$$\begin{aligned}
0 &= \mathrm{Re}\left\{\int_\Omega \overline{(\nabla - i\kappa H\mathbf{A})\widehat{\psi}}(\nabla - i\kappa H\mathbf{A})\psi + \overline{\widehat{\psi}}(|\psi|^2 - a)\psi\,dx\right\}\\
&\geq \mathrm{Re}\left\{\int_\Omega \overline{(\nabla - i\kappa H\mathbf{A})\widehat{\psi}}(\nabla - i\kappa H\mathbf{A})\psi + \overline{\widehat{\psi}}\left(|\psi|^2 - \overline{a}(\kappa)\right)\psi\,dx\right\}\\
&\geq \int_{\Omega_+} |\nabla|\psi||^2 + (|\psi| - \overline{a}(\kappa))\,|\psi||(\nabla - i\kappa H\mathbf{A})\widehat{\phi}|^2\\
&\qquad\qquad + \left(|\psi| + \sqrt{\overline{a}(\kappa)}\right)\left(|\psi| - \sqrt{\overline{a}(\kappa)}\right)^2|\psi|\,dx\,.
\end{aligned}$$

Since the integrand is non-negative in Ω_+, we easily conclude that Ω_+ has measure zero, and consequently, we get that $|\psi| \in L^\infty(\Omega)$.
Since Ω_+ has measure zero and $\psi \in C^0(\overline{\Omega})$, we get

$$|\psi(x)|^2 \leq \overline{a}(\kappa)\,, \qquad \forall x \in \overline{\Omega}\,.$$

□

Corollary 4.4.2. *Let $\kappa > 0$; If $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ is a critical point, we have,*

$$|\psi(x)|^2 \leq \max\{\overline{a}, 0\}\,, \qquad \forall x \in \overline{\Omega}\,, \tag{4.4.9}$$

where $\overline{a} = \sup_\kappa \overline{a}(\kappa)$ was introduced in (4.1.10).

The following estimates play an essential role in controlling the errors resulting from various approximations (see Section 4.5). These estimates are simpler than the delicate elliptic estimates in [15] and [35].

Proposition 4.4.3. *Suppose that* (4.1.15) *holds. Let $\beta \in (0, 1)$. There exist positive constants κ_0 and C such that, if $\kappa \geq \kappa_0$ and $(\psi, \mathbf{A})$ is a minimizer of* (4.1.1)*, then*

$$\|\operatorname{curl}(\mathbf{A} - \mathbf{F})\|_{L^2(\Omega)} \leq \frac{C}{H}\,. \tag{4.4.10}$$

$$\|\mathbf{A} - \mathbf{F}\|_{H^2(\Omega)} \leq \frac{C}{H}\,, \tag{4.4.11}$$

$$\|\mathbf{A} - \mathbf{F}\|_{C^{0,\beta}(\overline{\Omega})} \leq \frac{C}{H}\,. \tag{4.4.12}$$

Here we recall that $\mathbf{F}$ is the magnetic potential defined by

$$\operatorname{curl}\mathbf{F} = B_0\,,\ \mathbf{F} \in H^1_{\mathrm{div}}(\Omega)\,. \tag{4.4.13}$$

Proof. Under Assumption (4.1.15), Theorem 4.3.4 yields

$$\begin{aligned}\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)} &\leq \frac{1}{\kappa H}\,\mathrm{E_g}(\kappa,H,a,B_0)^{\frac12}\\ &\leq \frac{1}{\kappa H}\left(\kappa^2\int_{\{a(x,\kappa)>0\}} a(x,\kappa)^2\hat f\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)dx+\frac{\kappa^2}{2}\int_{\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\,dx\right)^{\frac12}.\end{aligned} \tag{4.4.14}$$

Using (4.1.6) and the bound $\hat f(b)\leq \frac12$, we get,

$$\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)} \leq \frac{C}{H}\,. \tag{4.4.15}$$

As in [6, Proposition 4.1], we prove that

$$\|\mathbf{A}-\mathbf{F}\|_{H^2(\Omega)} \leq \frac{C}{H}\,. \tag{4.4.16}$$

Now, the estimate in $C^{0,\beta}$-norm is a consequence of the continuous Sobolev embedding of $H^2(\Omega)$ in $C^{0,\beta}(\overline{\Omega})$. □

4.5 Lower bounds for the global and local energies

In this section, we suppose that $\mathcal{D}$ is an open set with smooth boundary such that $\overline{\mathcal{D}}\subset\Omega$ (or $\mathcal{D}=\Omega$). We will give a lower bound of the ground state energy $\mathrm{E_g}(\kappa,H,a,B_0)$ introduced in (4.1.14).

Proposition 4.5.1. *Under Assumptions* (4.1.4)-(4.1.7)*, there exist for all* $\beta\in(0,1)$ *positive constants* C *and* κ_0 *such that if* $\kappa\geq\kappa_0$, $\ell\in(0,\frac12)$, $\delta\in(0,1)$, $\rho>0$, $\ell^2\kappa H\rho>1$, $(\psi,\mathbf{A})$ *is a minimizer of* (4.1.1)*,* $h\in C^1(\overline{\Omega})$, $\|h\|_\infty\leq 1$ *and* $(\ell,x_0,\widetilde{x}_0)$ *is a* ρ*-admissible triple, then,*

$$\begin{aligned}\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(h\psi,\mathbf{A};a,Q_\ell(x_0)) &\geq (1-\delta)\kappa^2\left\{a_+(\widetilde{x}_0,\kappa)^2\hat f\left(\frac{H}{\kappa}\frac{|B_0(\widetilde{x}_0)|}{a_+(\widetilde{x}_0,\kappa)}\right)+\frac12 a_-(\widetilde{x}_0,\kappa)^2\right\}\\ &\quad - C\kappa^2\Big(\delta^{-1}\ell^2L(\kappa)^2+\delta^{-1}\kappa^2\ell^4+\delta^{-1}\ell^{2\beta}+(\kappa\ell)^{-1}+\ell\,L(\kappa)\Big)\,,\end{aligned} \tag{4.5.1}$$

where $L(\kappa)$ *is introduced in* (4.1.9).

Proof. We distinguish between two cases according to the sign of $a(\widetilde{x}_0,\kappa)$.
We begin with the case when $a(\widetilde{x}_0,\kappa)\leq 0$**.** We have,

$$\begin{aligned}\mathcal{E}_0(h\psi,\mathbf{A};a,Q_\ell(x_0)) &= \int_{Q_\ell(x_0)}|(\nabla-i\kappa H\mathbf{A})h\psi|^2\,dx+\frac{\kappa^2}{2}\int_{Q_\ell(x_0)}(a(x,\kappa)-|h\psi|^2)^2\,dx\\ &\geq \frac{\kappa^2}{2}\int_{Q_\ell(x_0)}a(x,\kappa)^2\,dx-\kappa^2\int_{Q_\ell(x_0)}a(x,\kappa)|h\psi|^2\,dx\,.\end{aligned}$$

Using (4.3.2), (4.4.9) and the assumptions on h, the simple decomposition $a(x,\kappa)=a(\widetilde{x}_0,\kappa)+$

$(a(x,\kappa) - a(\widetilde{x}_0,\kappa))$ yields for any $\delta \in (0,1)$

$$\begin{aligned}\frac{\kappa^2}{2}\int_{Q_\ell(x_0)} a(x,\kappa)^2\,dx &\geq (1-\delta)\frac{\kappa^2}{2}\int_{Q_\ell(x_0)} a(\widetilde{x}_0,\kappa)^2\,dx \\ &\qquad + (1-\delta^{-1})\frac{\kappa^2}{2}\int_{Q_\ell(x_0)} (a(x,\kappa)-a(\widetilde{x}_0,\kappa))^2\,dx \\ &\geq (1-\delta)\,\frac{\kappa^2}{2}\,a(\widetilde{x}_0,\kappa)^2\,|Q_\ell(x_0)| - C\delta^{-1}\kappa^2\ell^2 L(\kappa)^2\,|Q_\ell(x_0)|\,, \end{aligned} \tag{4.5.2}$$

and

$$\begin{aligned}-\kappa^2\int_{Q_\ell(x_0)} a(x,\kappa)|h\psi|^2\,dx &\geq -\kappa^2\int_{Q_\ell(x_0)} a(\widetilde{x}_0,\kappa)|h\psi|^2\,dx - C\,\ell\,L(\kappa)\,\kappa^2\,|Q_\ell(x_0)| \\ &\geq -C\,\ell\,L(\kappa)\,\kappa^2\,|Q_\ell(x_0)|\,. \end{aligned} \tag{4.5.3}$$

Collecting (4.5.2) and (4.5.3), we get,

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(h\psi,\mathbf{A};a,Q_\ell(x_0)) \geq (1-\delta)\,\frac{\kappa^2}{2}\,a(\widetilde{x}_0,\kappa)^2 - C\delta^{-1}\kappa^2\ell^2 L(\kappa)^2 - C'\,\ell\,L(\kappa)\,\kappa^2\,. \tag{4.5.4}$$

Now, we treat the case when $a(\widetilde{x}_0,\kappa) > 0$**.** Let $\phi_{x_0}(x) = (\mathbf{A}(x_0) - \mathbf{F}(x_0))\cdot x$, where $\mathbf{F}$ is the magnetic potential introduced in (4.4.13). Using the estimate of $\|\mathbf{A}-\mathbf{F}\|_{C^{0,\beta}(\Omega)}$ given in Proposition 4.4.3, we get for any $\beta \in (0,1)$ the existence of a constant C such that for all $x \in Q_\ell(x_0)$,

$$|\mathbf{A}(x) - \nabla\phi_{x_0} - \mathbf{F}(x)| \leq C\,\frac{\ell^\beta}{H}\,. \tag{4.5.5}$$

Let $\widetilde{x}_0 \in \overline{Q_\ell(x_0)}$ and $\varphi = \varphi_{x_0,\widetilde{x}_0} + \phi_{x_0}$ with $\varphi_{x_0,\widetilde{x}_0}$ satisfying (4.3.4). We define the function in $Q_\ell(x_0)$,

$$u(x) = e^{-i\kappa H\varphi}h\psi(x)\,. \tag{4.5.6}$$

Similarly to (4.3.9), we have, for any $\delta \in (0,1)$,

$$\frac{\kappa^2}{2}\int_{Q_\ell(x_0)}\left(a(x,\kappa) - |h\psi|^2\right)^2\,dx \geq (1-\delta)\frac{\kappa^2}{2}\int_{Q_\ell(x_0)}\left(a(\widetilde{x}_0,\kappa)-|h\psi|^2\right)^2\,dx - C\delta^{-1}\kappa^2\ell^4 L(\kappa)^2\,. \tag{4.5.7}$$

Using the same techniques as in [5, Lemma 4.1], we get, for any $\beta \in (0,1)$,

$$\begin{aligned}\int_{Q_\ell(x_0)}|(\nabla - i\kappa H\mathbf{A})h\psi|^2\,dx &\geq (1-\delta)\int_{Q_\ell(x_0)}|(\nabla - i\kappa H(\zeta_\ell|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0)+\nabla\varphi(x)))h\psi|^2\,dx \\ &\qquad - C\delta^{-1}(\kappa H)^2\left(\ell^4 + \frac{\ell^{2\beta}}{H^2}\right)\int_{Q_\ell(x_0)}|h\psi|^2\,dx\,. \end{aligned} \tag{4.5.8}$$

Thus, by collecting (4.5.7) and (4.5.8), using (4.1.7), (4.4.9) and $\|h\|_{L^\infty(\Omega)} \leq 1$, we get

$$\begin{aligned}\mathcal{E}_0(h\psi, \mathbf{A}; a(\widetilde{x}_0, \kappa), Q_\ell(x_0)) \geq (1-\delta)\mathcal{E}_0(e^{-i\kappa H\varphi}h\psi(x), \zeta_\ell |B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0); a(\widetilde{x}_0, \kappa), Q_\ell(x_0)) \\ - C\delta^{-1}\kappa^2\ell^4 L(\kappa)^2 - C_1\delta^{-1}\kappa^2 H^2\left(\ell^4 + \frac{\ell^{2\beta}}{H^2}\right)\ell^2\,. \quad (4.5.9)\end{aligned}$$

Let R and b be as in (4.3.8). Let us introduce the function $v_{\ell,x_0,\widetilde{x}_0}$ in Q_R as follows :

$$v_{\ell,x_0,\widetilde{x}_0}(x) = \begin{cases} u\left(\frac{\ell}{R}x + x_0\right) & \text{if } x \in Q_R \subset \{B_0 > \rho\} \cap \Omega \\ \overline{u}\left(\frac{\ell}{R}x + x_0\right) & \text{if } x \in Q_R \subset \{B_0 < -\rho\} \cap \Omega\,, \end{cases} \quad (4.5.10)$$

where u is defined in (4.5.6).
Similarly to (4.3.12), we use the change of variable $y = \frac{R}{\ell}(x - x_0)$ and get

$$\mathcal{E}_0(e^{-i\kappa H\varphi}h\psi(x), \zeta_\ell\, \kappa H|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0); a(\widetilde{x}_0, \kappa), Q_\ell(x_0)) = \frac{1}{b}F^{+1,a(\widetilde{x}_0,\kappa)}_{b,Q_R}(v_{\ell,x_0,\widetilde{x}_0})\,, \quad (4.5.11)$$

where $F^{+1,a(\widetilde{x}_0,\kappa),}_{b,Q_R}$ is introduced in (4.2.1).
Since $v_{\ell,x_0,\widetilde{x}_0} \in H^1(Q_R)$ then, using (4.2.12) and (4.2.13), we get

$$\begin{aligned}\frac{1}{b}F^{+1,a(\widetilde{x}_0,\kappa)}_{b,Q_R}(v_{\ell,x_0,\widetilde{x}_0}) &\geq \frac{1}{b}e_N\left(b, R, a(\widetilde{x}_0,\kappa)\right) \\ &\geq \frac{1}{b}e_D\left(b, R, a(\widetilde{x}_0,\kappa)\right) - C_M\, a(\widetilde{x}_0,\kappa)^{\frac{3}{2}}\frac{R}{\sqrt{b}} \\ &\geq a(\widetilde{x}_0,\kappa)^2\frac{R^2}{b}\hat{f}\left(\frac{b}{a(\widetilde{x}_0,\kappa)}\right) - \widehat{C}_M\frac{R}{\sqrt{b}}\,. \end{aligned} \quad (4.5.12)$$

Inserting (4.5.12) into (4.5.11), we get

$$\begin{aligned}\mathcal{E}_0(e^{-i\kappa H\varphi}h\psi(x), \zeta_\ell\, \kappa H|B_0(\widetilde{x}_0)|\mathbf{A}_0(x-x_0); a(\widetilde{x}_0, \kappa), Q_\ell(x_0)) \geq a(\widetilde{x}_0,\kappa)^2\frac{R^2}{b}\hat{f}\left(\frac{b}{a(\widetilde{x}_0,\kappa)}\right) \\ - \widehat{C}_M\frac{R}{\sqrt{b}}\,. \quad (4.5.13)\end{aligned}$$

Having in mind (4.3.8) and (4.5.13), we get from (4.5.9),

$$\begin{aligned}\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0(h\psi, \mathbf{A}; a(\widetilde{x}_0, \kappa), Q_\ell(x_0)) \geq (1-\delta)\kappa^2 a(\widetilde{x}_0,\kappa)^2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(\widetilde{x}_0)|}{a(\widetilde{x}_0,\kappa)}\right) \\ - C\delta^{-1}\kappa^2\ell^2 L(\kappa)^2 - C_1\delta^{-1}\kappa^2 H^2\left(\ell^4 + \frac{\ell^{2\beta}}{H^2}\right) - C_2\frac{\kappa}{\ell}\,. \quad (4.5.14)\end{aligned}$$

The estimates in (4.5.4) and (4.5.14) achieve the proof of Proposition 4.5.1. ☐

Application 4.5.2. *We keep the same choice of ℓ, ρ, $L(\kappa)$ and δ as in* (4.3.18), (4.3.19) *and choose :*

$$\beta = \frac{3}{4}\,. \quad (4.5.15)$$

This choice and Assumption (4.1.15) *permit to have the assumptions in Proposition 4.5.1 satisfied and make the error terms in its statement of order* $o(\kappa^2)$*. We have as* $\kappa \longrightarrow \infty$*,*

$$\delta^{-1}\kappa^4\ell^4 = \kappa^{\frac{21}{12}} \ll \kappa^2\,,$$

$$\delta^{-1}\kappa^2\ell^{2\beta} = \kappa^{\frac{29}{24}} \ll \kappa^2\,,$$

$$\delta^{-1}\kappa^2\ell^2 L(\kappa)^2 = \kappa^{\frac{23}{12}} \ll \kappa^2\,,$$

$$\frac{\kappa}{\ell} = \kappa^{\frac{19}{12}} \ll \kappa^2\,,$$

$$\ell\, L(\kappa)\,\kappa^2 = \kappa^{\frac{23}{12}} \ll \kappa^2\,,$$

$$\ell^2\kappa H\rho = \kappa^{\frac{3}{24}} \gg 1\,.$$

The next theorem presents a lower bound of the local energy in a relatively compact smooth domain $\mathcal{D}$ in Ω. We deduce the lower bound of the global energy by replacing $\mathcal{D}$ by Ω.

Theorem 4.5.3.
Under Assumptions (4.1.4)-(4.1.8)*, if* (4.1.15) *holds,* $L(\kappa) \le C\,\kappa^{\frac{1}{2}}$ *with* $C > 0$*,* $h \in C^1(\overline{\Omega})$*,* $\|h\|_\infty \le 1$*,* $(\psi, \mathbf{A})$ *is a minimizer of* (4.1.1) *and* $\mathcal{D}$ *an open set in* Ω*, then as* $\kappa \longrightarrow +\infty$*,*

$$\mathcal{E}(h\psi, \mathbf{A}; a, B_0, \mathcal{D}) \ge \mathcal{E}_0(h\psi, \mathbf{A}; a, \mathcal{D}) \ge \kappa^2 \int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2 \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) dx + \frac{\kappa^2}{2}\int_{\mathcal{D}\cap\{a(x,\kappa)\le 0\}} a(x,\kappa)^2\, dx + o\left(\kappa^2\right)\,. \quad (4.5.16)$$

Proof. The proof is similar to the one in Theorem 4.3.4 and we keep the same notation. Let

$$\mathcal{D}^+_{\ell,\rho} = \mathrm{int}\left(\cup_{\gamma\in\mathcal{I}^+_{\ell,\rho}}\overline{Q_{\gamma,\ell}}\right) \qquad \text{and} \qquad \mathcal{D}^-_{\ell,\rho} = \mathrm{int}\left(\cup_{\gamma\in\mathcal{I}^-_{\ell,\rho}}\overline{Q_{\gamma,\ell}}\right)\,,$$

where $\gamma \in \mathcal{I}^+_{\ell,\rho}$ and $\gamma \in \mathcal{I}^-_{\ell,\rho}$ are introduced in (4.3.21).
Thanks to Proposition 4.5.1, we can easily prove the existence of positive constant C such that for any $\delta \in (0,1)$ and $\beta \in (0,1)$,

$$\mathcal{E}_0(h\psi, \mathbf{A}; a, \mathcal{D}) \ge \kappa^2(1-\delta)\left\{\int_{\mathcal{D}^+_{\ell,\rho}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)dx + \frac{1}{2}\int_{\mathcal{D}^-_{\ell,\rho}\cap\{a(x,\kappa)\le 0\}} a(x,\kappa)^2\,dx\right\} - C\,r(\kappa,\ell,\delta,\rho,L(\kappa),\beta)\,,$$

where

$$r(\kappa,\ell,\delta,\rho,L(\kappa),\beta) = \kappa^2\ell + \kappa^2\rho + \frac{\kappa}{\ell} + \delta^{-1}\kappa^2\ell^2L(\kappa)^2 + \delta^{-1}\kappa^4\ell^4 + \delta^{-1}\kappa^2\ell^{2\beta} + \ell\, L(\kappa)\,\kappa^2\,. \quad (4.5.17)$$

Notice that using the regularity of $\partial\mathcal{D}$, (4.1.4) and (4.1.8) (see (4.3.1)), we get the existence of constants $C_1 > 0$ and $C_2 > 0$ such that,

$$\forall \ell \leq C_2\, \kappa^{-\frac{1}{2}}\,, \quad \forall \rho \in (0,1)\,, \qquad |\mathcal{D} \setminus \mathcal{D}^+_{\ell,\rho}| + |\mathcal{D} \setminus \mathcal{D}^-_{\ell,\rho}| \leq C_1(\kappa^{\frac{1}{2}}\,\ell + \rho)\,. \tag{4.5.18}$$

This implies by using (4.1.7) and the upper bound $\hat{f} \leq \frac{1}{2}$,

$$\int_{\mathcal{D}^+\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2 \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) dx \geq \int_{\mathcal{D}^+_{\ell,\rho}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2 \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) dx - \frac{1}{2}\,\overline{a}\,|\mathcal{D}\setminus\mathcal{D}_{\ell,\rho}| \tag{4.5.19}$$

and

$$\frac{1}{2}\int_{\mathcal{D}^-\cap\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\,dx \geq \frac{1}{2}\int_{\mathcal{D}_{\ell,\rho}\cap\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\,dx - \frac{1}{2}\,\overline{a}\,|\mathcal{D}\setminus\mathcal{D}^-_{\ell,\rho}|\,, \tag{4.5.20}$$

where $\overline{a}$ is introduced in (4.1.10).
Collecting (4.5.19) and (4.5.20), using Assumptions (4.1.6) and (4.5.18), we find that,

$$\mathcal{E}_0(h\psi,\mathbf{A};a,\mathcal{D}) \geq \kappa^2(1-\delta)\left\{\int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)dx + \frac{1}{2}\int_{\mathcal{D}\cap\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\,dx\right\} - C\,\hat{r}(\kappa,\ell,\delta,\rho,L(\kappa),\beta)\,, \tag{4.5.21}$$

where $\hat{r}(\kappa,\ell,\delta,\rho,L(\kappa),\beta)$ satisfies (4.5.17).
Under Assumption (4.1.15), the choice of the parameters ρ, ℓ, $L(\kappa)$ in (4.3.18), δ in (4.3.19) and β in (4.5.15), implies that all error terms are of lower order compared to κ^2.
As a consequence of (4.1.15), the inequality (4.5.21) becomes as $\kappa \longrightarrow +\infty$

$$\mathcal{E}_0(h\psi,\mathbf{A};a,\mathcal{D}) \geq \kappa^2\left\{\int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)dx + \frac{1}{2}\int_{\mathcal{D}\cap\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\,dx\right\} + o(\kappa^2)\,. \tag{4.5.22}$$

Moreover, we know that

$$\mathcal{E}(h\psi,\mathbf{A};a,B_0,\mathcal{D}) \geq \mathcal{E}_0(h\psi,\mathbf{A};a,\mathcal{D})\,.$$

This achieves the proof of Theorem 4.5.3. □

As we now show, Theorem 4.5.3 permits to achieve the proof of two statements presented in the introduction :

Proof of Corollary 4.1.3.
If $(\psi,\mathbf{A})$ is a minimizer of (4.1.1), we have

$$\mathrm{E_g}(\kappa,H) = \mathcal{E}_0(\psi,\mathbf{A};a,\Omega) + (\kappa H)^2\int_\Omega |\operatorname{curl}(\mathbf{A}-\mathbf{F})|^2\,dx\,, \tag{4.5.23}$$

where $\mathcal{E}_0(\psi, \mathbf{A}; a, \Omega)$ is defined in (4.1.19).
Using (4.1.17) and (4.5.22) (with $\mathcal{D} = \Omega$), then under Assumption (4.1.15) as $\kappa \longrightarrow +\infty$

$$\mathcal{E}_0(\psi, \mathbf{A}; a, \Omega) = \kappa^2 \int_{\{a(x,\kappa)>0\}} a(x,\kappa)^2 \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) dx + \frac{\kappa^2}{2}\int_{\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\, dx + o\left(\kappa^2\right). \tag{4.5.24}$$

Putting (4.5.24) and (4.1.17) into (4.5.23), we finish the proof of Corollary 4.1.3. □

Proof of Theorem 4.1.4.
Noticing that (4.5.22) is valid when $h = 1$ and $\mathcal{D}$ replaced by $\overline{\mathcal{D}}^c := \Omega \setminus \overline{\mathcal{D}}$ for any open domain $\mathcal{D} \subset \Omega$ with smooth boundary, then we get :

$$\mathcal{E}_0(\psi, \mathbf{A}; a, \overline{\mathcal{D}}^c) \geq \kappa^2 \left\{ \int_{\overline{\mathcal{D}}^c \cap \{a(x,\kappa)>0\}} a(x,\kappa)^2 \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) dx \right.$$
$$\left. + \frac{1}{2}\int_{\overline{\mathcal{D}}^c \cap \{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\, dx \right\} + o(\kappa^2). \tag{4.5.25}$$

We can decompose $\mathcal{E}_0(\psi, \mathbf{A}; a, \mathcal{D})$ as follow :

$$\mathcal{E}_0(\psi, \mathbf{A}; a, \mathcal{D}) = \mathcal{E}_0(\psi, \mathbf{A}; a, \Omega) - \mathcal{E}_0(\psi, \mathbf{A}; a, \overline{\mathcal{D}}^c).$$

Using (4.5.24) and (4.5.25), we get

$$\mathcal{E}_0(\psi, \mathbf{A}; a, \mathcal{D}) \leq \kappa^2 \left\{ \int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2 \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) dx + \frac{1}{2}\int_{\mathcal{D}\cap\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\, dx \right\}$$
$$+ o(\kappa^2). \tag{4.5.26}$$

□

4.6 Study of examples

In this section, we will describe situations where the remainder term in (4.1.17) is indeed small as $\kappa \to +\infty$ compared with the leading order term

$$E_{\mathrm{g}}^{\mathbf{L}}(\kappa, H, a, B_0) := \kappa^2 \left(\int_{\{a(x,\kappa)>0\}} a(x,\kappa)^2 \hat{f}\left(\sigma\frac{|B_0(x)|}{a(x,\kappa)}\right) dx + \frac{1}{2}\int_{\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\, dx \right), \tag{4.6.1}$$

where,

$$\sigma = \frac{H}{\kappa}. \tag{4.6.2}$$

Note that $0 < \lambda_{\min} \leq \sigma \leq \lambda_{\max}$, so that σ will be considered as an independent parameter in $[\lambda_{\min}\,,\,\lambda_{\max}]$.
We will also explore, case by case how one can verify Assumption (A_4) as formulated precisely in (4.3.1).

Proposition 4.6.1. *Suppose* (4.1.4) *and* (4.1.15) *hold. Let* $a(x,\kappa) = a(x)$ *where* $a(x) \in C^1(\overline{\Omega})$ *is a function independent of* κ *and satisfies,*

$$\begin{cases} \{x \in \Omega :\, a(x) > 0\} \neq \varnothing\,, \\ \text{or} \\ \{x \in \Omega :\, a(x) < 0\} \neq \varnothing\,. \end{cases} \tag{4.6.3}$$

There exist positive constants C *and* κ_0 *such that,*

$$\forall \kappa \geq \kappa_0\,, \qquad E_{\rm g}^{\mathbf{L}}(\kappa, H, a, B_0) \geq C\,\kappa^2\,.$$

Proof. Since $a(x,\kappa) = a(x)$, the energy $E_{\rm g}^{\mathbf{L}}$ becomes :

$$E_{\rm g}^{\mathbf{L}}(\kappa, H, a, B_0) := \kappa^2 \left(\int_{\{a(x)>0\}} a(x)^2\, \hat{f}\left(\sigma \frac{|B_0(x)|}{a(x)}\right)\, dx + \frac{1}{2} \int_{\{a(x)\leq 0\}} a(x)^2\, dx \right)\,.$$

Each term being positive, it is clear that the leading term is positive if $\{x \in \Omega :\, a(x) < 0\} \neq \varnothing$. If $\{x \in \Omega :\, a(x) < 0\} = \varnothing$ and $\{x \in \Omega :\, a(x) > 0\} \neq \varnothing$, there exist $\rho_0 > 0$, $a_0 > 0$ and a disk $D(x_0, r_0)$ such that

$$D(x_0, r_0) \subset \{a(x) > a_0\} \cap \{|B_0| > \rho_0\}\,.$$

Using the monotonicity of $\hat{f}$ and the bound of $a(x)$ in (4.1.6), we may write

$$\begin{aligned} \int_{\{a(x)>0\}} a(x)^2\, \hat{f}\left(\frac{H}{\kappa} \frac{|B_0(x)|}{a(x)}\right)\, dx &\geq \int_{D(x_0,r_0)} a(x)^2\, \hat{f}\left(\sigma \frac{|B_0(x)|}{a(x)}\right)\, dx \\ &\geq \pi\, r_0^2\, a_0^2\, \hat{f}\left(\frac{\rho_0}{\overline{a}} \sigma\right)\,, \end{aligned} \tag{4.6.4}$$

where $\overline{a}$ is introduced in (4.1.10).
In particular, when (4.1.15) is satisfied, there exists $\kappa_0 > 0$ such that

$$\forall \kappa \geq \kappa_0\,, \qquad \int_{\{a(x)>0\}} a(x)^2\, \hat{f}\left(\frac{H}{\kappa} \frac{|B_0(x)|}{a(x)}\right)\, dx \geq \pi\, r_0^2\, a_0^2\, \hat{f}\left(\frac{\rho_0}{\overline{a}} \lambda_{\min}\right)\,. \tag{4.6.5}$$

□

Proposition 4.6.2 (Verification of (A_4)). *Suppose that the function* a *satisfies (see Fig.4.1),*

$$\begin{cases} |a| + |\nabla a| > 0 & \text{in } \overline{\Omega}\,, \\ \nabla a \times \vec{n} \neq 0 & \text{on } \widetilde{\Gamma} \cap \partial\Omega\,, \end{cases} \tag{4.6.6}$$

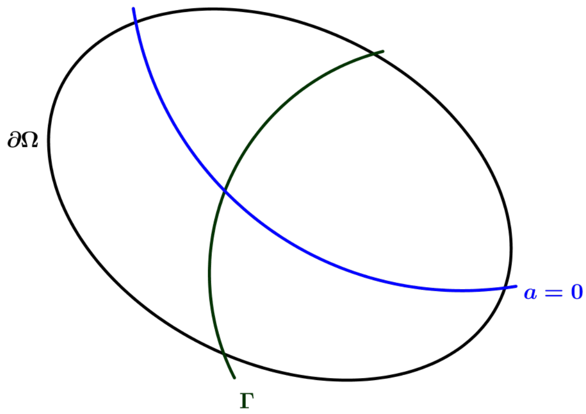

FIGURE 4.1 – Schematic representation of Ω with pinning term independent of κ and with variable magnetic field.

where $\widetilde{\Gamma}$ *defined as follows :*

$$\widetilde{\Gamma} = \{x \in \overline{\Omega} : a(x) = 0\}\,. \tag{4.6.7}$$

Then Assumption (A_4) *is satisfied.*

Proof. From (4.6.6), we observe that,

$$\mathrm{card}\,\{\gamma \in \Gamma_\ell \cap \Omega \text{ with } Q_\ell(\gamma) \cap \partial\{a > 0\} \neq \emptyset\} = \mathrm{card}\,\{\gamma \in \Gamma_\ell \cap \Omega \text{ with } Q_\ell(\gamma) \cap \widetilde{\Gamma} \neq \emptyset\}\,.$$

Let $\epsilon \in (0,1)$, we introduce the domain

$$D_\epsilon = \{x \in \Omega : \mathrm{dist}(x, \widetilde{\Gamma}) \leq \epsilon\}\,.$$

Now we give a rough upper bound for the area of D_ϵ.
By assumption $\widetilde{\Gamma}$ consists of a finite number of connected curves, which are either closed in Ω or join two points of $\partial\Omega$. Let us consider the first case, we denote by $\widetilde{\Gamma}^{(1)}$ such a curve. We can parametrize this curve using the standard tubular coordinates (s,t), where s measures the arc-length in $\widetilde{\Gamma}^{(1)}$ and t measures the distance to $\widetilde{\Gamma}^{(1)}$ (see [14, Appendix F] for the detailed construction of these coordinates).

In the neighborhood of $\widetilde{\Gamma}^{(1)}$, we choose one point γ_0 on $\widetilde{\Gamma}^{(1)}$ corresponding to $(0,0)$. Let $N \in \mathbb{N}$ and $\mathcal{L}$ the length of $\widetilde{\Gamma}^{(1)}$. We consider for $i = 0, ..., N$, $s_i = \frac{i}{N}\mathcal{L}$ (modulo $\mathcal{L}\mathbb{Z}$) and $\gamma_i = (s_i, 0)$.
Notice that, there exists a positive constant C such that,

$$|\,\mathrm{dist}(\gamma_i, \gamma_{i+1})| = (1+\epsilon_i)|s_i - s_{i+1}|\,, \qquad \left(-\frac{C}{N} \leq \epsilon_i < 0\right).$$

Thus,

$$\left|\left\{x\in\Omega:\operatorname{dist}\left(x,\widetilde{\Gamma}^{(1)}\right)\leq\frac{\mathcal{L}}{N}\right\}\right|\leq\sum_i\left|Q_{\frac{\mathcal{L}}{N}}((s_i,0))\right|\,.$$

Coming back to our problem, we select $N=\left[\frac{\mathcal{L}}{\epsilon}\right]$ and we note that

$$\frac{\mathcal{L}}{N+1}\leq\epsilon\leq\frac{\mathcal{L}}{N}\,,$$

which implies that,

$$\begin{aligned}|D_\epsilon|&\leq\frac{\mathcal{L}^2}{N}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right)\\&\leq\mathcal{L}\,\epsilon\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right)=\epsilon\mathcal{L}(1+\mathcal{O}(\epsilon))\,.\end{aligned}$$

Hence we have shown that,

$$\limsup_{\epsilon\to0}\frac{|D_\epsilon|}{\epsilon}\leq\mathcal{L}\,.$$

In a similar fashion, we prove that

$$\liminf_{\epsilon\to0}\frac{|D_\epsilon|}{\epsilon}\geq\mathcal{L}\,.$$

and, as a consequence, we end up with the following conclusion :

$$\lim_{\epsilon\to0}\frac{|D_\epsilon|}{\epsilon}=\mathcal{L}\,.\tag{4.6.8}$$

Coming back to Assumption (A_4), we now observe that all the $Q_\ell(\gamma)$ touching $\widetilde{\Gamma}$ are inside $D_{\sqrt{2}\ell}$, hence we get, by comparison of the area

$$\ell^2\text{card}\,\{\gamma\in\Gamma_\ell\cap\Omega\text{ with }Q_\ell(\gamma)\cap\widetilde{\Gamma}\neq\emptyset\}\leq C\,\ell\,,$$

and consequently, there exist positive constants C_1, C_2 and κ_0 such that

$$\forall\kappa\geq\kappa_0\,,\,\forall\ell\leq C_2\kappa^{-\frac12}\,,\,\text{card}\,\{\gamma\in\Gamma_\ell\cap\Omega\text{ with }Q_\ell(\gamma)\cap\partial\{a>0\}\neq\emptyset\}\leq C_1\,\ell^{-1}\,,$$

which is a stronger form of (A_4). □

4.6.1 The case with a κ-dependent oscillation.

4.6.1.1 Preliminaries

We start with two lemmas which are standard in homogenization theory (see [10, Section 16-17])

Lemma 4.6.3. *Let $D\subset\mathbb{R}^2$ be a bounded open set and φ be a Γ_{T_1,T_2}-periodic continuous function* *in* $\mathbb{R}^2$ *with* $\Gamma_{T_1,T_2} = T_1\mathbb{Z}\times T_2\mathbb{Z}$. *There exists a positive constant* M_0 *such that if* $M \geq M_0$, *then,*

$$\int_D \varphi(Mx)\,dx = \frac{|D|}{T_1T_2}\int_0^{T_1}\int_0^{T_2}\varphi(t_1,t_2)dt_1dt_2 + \mathcal{O}(M^{-1}).$$

Lemma 4.6.4. *Let* $D\subset\mathbb{R}^2$ *be a bounded open set and* $\phi:\mathbb{R}^2\times\overline{D}\longrightarrow\mathbb{R}^2$ *be a continuous function satisfying :*

$$\phi(t+T,x) = \phi(t,x)\,,\qquad \forall T\in T_1\mathbb{Z}\times T_2\mathbb{Z}\,, \tag{4.6.9}$$

and uniformly Lipschitz, i.e. with the property that there exist constants $C>0$ *and* ϵ_0, *such that,*

$$|\phi(t,x)-\phi(t,\widetilde{x})| \leq C\,|x-\widetilde{x}|\,,\quad \forall t\in\mathbb{R}^2\,,\,\forall x,\widetilde{x}\in\overline{D},\ \text{s.t.}\,|x-\widetilde{x}|<\epsilon_0\,. \tag{4.6.10}$$

There exists a positive constant M_0 *such that if* $M\geq M_0$, *then,*

$$\int_D \phi(Mx,x)\,dx = \int_D \overline{\phi}(x)\,dx + \mathcal{O}(M^{-1})\,,$$

where,

$$\overline{\phi}(x) = \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}\phi((t_1,t_2),x)\,dt_1dt_2\,. \tag{4.6.11}$$

4.6.1.2 First example :

Proposition 4.6.5. *Suppose that* (4.1.4) *and* (4.1.15) *hold. Let* $a(x,\kappa)=\alpha(\kappa^{\frac{1}{2}}\,x)$ *where* $\alpha(\cdot)\in C^1(\overline{\Omega})$ *is a* Γ_{T_1,T_2}*-periodic function*[1]. *Then the leading order term* $E_{\rm g}^{\boldsymbol{L}}$ *defined in* (4.6.1) *satisfies,*

$$E_{\rm g}^{\boldsymbol{L}}(\kappa,H,a,B_0) = \kappa^2\int_\Omega \overline{\phi}_+(x)\,dx + \kappa^2|\Omega|\,\overline{\phi}_- + o(\kappa^2)\,,\quad \text{as } \kappa\to+\infty\,.$$

Here,

$$\overline{\phi}_+(x) = \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}\alpha_+(t_1,t_2)^2\,\hat{f}\left(\sigma\frac{|B_0(x)|}{\alpha_+(t_1,t_2)}\right)\,dt_1dt_2\,,$$

and

$$\overline{\phi}_- = \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}\alpha_-(t_1,t_2)^2\,dt_1dt_2\,.$$

Proof.
We first estimate the second term in (4.6.1)**.** We apply Lemma 4.6.3 with $D=\Omega$, $M=\kappa^{\frac{1}{2}}$ and $\varphi=\alpha_-^2$, we obtain,

$$\int_\Omega a_-(x,\kappa)^2\,dx = \frac{|\Omega|}{T_1T_2}\int_0^{T_1}\int_0^{T_2}\alpha_-(t_1,t_2)^2\,dt_1dt_2 + \mathcal{O}(\kappa^{-\frac{1}{2}})\,,$$

[1] see Fig. 4.2

and consequently,

$$\kappa^2 \int_{\{a(x)\leq 0\}} a(x,\kappa)^2\,dx = \kappa^2\,\frac{|\Omega|}{T_1T_2}\int_0^{T_1}\int_0^{T_2}\alpha_-(t_1,t_2)^2\,dt_1dt_2 + \mathcal{O}(\kappa^{\frac{3}{2}})\,.$$

Now, we estimate the first term in (4.6.1)**.** We first prove that $\hat{f}$ is a Lipschitz function in $[\mathfrak{b}_0,1]$ with $\mathfrak{b}_0\in(0,1)$. We consider this restriction because when $\mathfrak{b}\to 0_+$ (see [6, Theorem 2.1]), $\hat{f}$ satisfies,

$$\hat{f}(\mathfrak{b}) = \frac{\mathfrak{b}}{2}\ln\frac{1}{\mathfrak{b}}(1+o(1))\,, \tag{4.6.12}$$

and $\hat{f}$ is not a Lipschitz function at 0. We recall the definition of $\hat{f}$

$$\hat{f}\,(\mathfrak{b}) = \lim_{R\longrightarrow\infty}\frac{e_D(\mathfrak{b},R)}{R^2}\qquad (\forall \mathfrak{b}\in[0,1])\,,$$

where

$$e_D(\mathfrak{b},R) = \inf_u F^{+1,+1}_{\mathfrak{b},Q_R}(u) := \inf_u \int_{Q_R}\left(\mathfrak{b}|(\nabla - i\mathbf{A}_0)u|^2 + \frac{1}{2}\left(1-|u|^2\right)^2\right)dx\,.$$

From the definition, we can conclude that $\hat{f}$ is concave and hence locally Lipschitz in $(0,+\infty)$ (see [22, Theorem 2.35]). For completion we write below a proof making explicit the Lipschitz constant. For $\mathfrak{b}'>0$, let $u_{\mathfrak{b}',R}\in H^1_0(Q_R)$ be a minimizer of $F^{+1,+1}_{\mathfrak{b}',Q_R}$. Then for all $\mathfrak{b}\in(0,1)$, we have,

$$e_D(\mathfrak{b},R)\leq F^{+1,+1}_{\mathfrak{b},Q_R}(u_{\mathfrak{b}',R})\leq e_D(\mathfrak{b}',R) + \|(\nabla - i\mathbf{A}_0)u_{\mathfrak{b}',R}\|^2_{L^2(Q_R)}|\mathfrak{b}-\mathfrak{b}'|\,.$$

Now, we estimate $\|(\nabla - i\mathbf{A}_0)u_{\mathfrak{b}',R}\|^2_{L^2(Q_R)}$ from above. Coming back to the definition, we get the existence of a positive constant C, such that for any $\mathfrak{b}\in[\mathfrak{b}_0,1]$ and for any $\mathfrak{b}'\in[\mathfrak{b}_0,1]$,

$$\|(\nabla - i\mathbf{A}_0)u_{\mathfrak{b}',R}\|^2_{L^2(Q_R)}\leq \frac{e_D(\mathfrak{b}',R)}{\mathfrak{b}'}\,.$$

This implies that,

$$e_D(\mathfrak{b},R)\leq e_D(\mathfrak{b}',R) + \frac{e_D(\mathfrak{b}',R)}{\mathfrak{b}'}|\mathfrak{b}-\mathfrak{b}'|\,.$$

Dividing by R^2 and taking the limit as $R\to+\infty$, we obtain

$$\hat{f}(\mathfrak{b})\leq \hat{f}(\mathfrak{b}') + \frac{|\hat{f}(\mathfrak{b}')|}{\mathfrak{b}'}|\mathfrak{b}-\mathfrak{b}'|\,.$$

Using the asymptotic behavior of $\hat{f}$ in (4.6.12) as $\mathfrak{b}'\to 0_+$, we finally obtain the existence of C such that

$$\hat{f}(\mathfrak{b})\leq \hat{f}(\mathfrak{b}') + C\left(\log\frac{1}{\mathfrak{b}_0}\right)|\mathfrak{b}-\mathfrak{b}'|\,,\ \forall \mathfrak{b},\mathfrak{b}' \text{ with } 1>\mathfrak{b}>\mathfrak{b}_0 \text{ and } 1>\mathfrak{b}'>\mathfrak{b}_0\,.$$

Exchanging $\mathfrak{b}$ and $\mathfrak{b}'$, we have proved the

Lemma 4.6.6. *$\hat{f}$ is locally Lipschitz in $(0,+\infty)$. More precisely, there exists C such that for* *any* $\mathfrak{b}_0 > 0$,

$$|\hat{f}(\mathfrak{b}) - \hat{f}(\mathfrak{b}')| \leq C \left(\log \frac{1}{\mathfrak{b}_0}\right) |\mathfrak{b} - \mathfrak{b}'|\,, \forall \mathfrak{b}, \mathfrak{b}' \textit{ with } 1 > \mathfrak{b} > \mathfrak{b}_0 \textit{ and } 1 > \mathfrak{b}' > \mathfrak{b}_0\,. \tag{4.6.13}$$

In addition, we have

$$|\hat{f}(\mathfrak{b}) - \hat{f}(\mathfrak{b}')| \leq 2\,|\mathfrak{b} - \mathfrak{b}'|\,, \forall \mathfrak{b}, \mathfrak{b}' \textit{ with } \mathfrak{b} > \frac{1}{2} \textit{ and } \mathfrak{b}' > \frac{1}{2}\,. \tag{4.6.14}$$

To continue, we consider

$$\mathbb{R}^2 \times \Omega_\rho \ni (t,x) \mapsto \phi(t,x) = \alpha_+(t)^2\, \hat{f}\left(\sigma \frac{|B_0(x)|}{\alpha_+(t)}\right)\,,$$

where, $\Omega_\rho := \Omega \cap \{|B_0| > \rho\}$.
The periodicity condition in (4.6.9) is clear. Let us verify the Lipschitz property. Let

$$\mathfrak{b}_0 = \frac{\lambda_{\min}}{\alpha_0}\,\rho\,,$$

where, $\lambda_{\min}$ is introduced in (4.1.15) and $\alpha_0 = \sup \alpha_+(t)$.

Let $\epsilon > 0$, $\mathcal{I}_+ = \{t \in \mathbb{R} : \alpha_+(t) \geq \epsilon\}$ and $\mathcal{I}_- = \{t \in \mathbb{R} : \alpha_+(t) \leq \epsilon\}$, we distinguish between two cases :
Case 1 : $(\alpha_+(t) \geq \epsilon)$. We observe that for $(x,t) \in \Omega_\rho \times \mathcal{I}_+$, we have

$$\mathfrak{b}_0 \leq \sigma \frac{|B_0(x)|}{\alpha_+(t)} \leq \frac{\sigma\,|B_0(x)|}{\epsilon}\,.$$

Thus, for any $t \in \mathcal{I}_+$ and for any $x, x' \in \overline{\Omega}_\rho$, we get

$$\begin{aligned}\left|\alpha_+(t)^2\, \hat{f}\left(\sigma \frac{|B_0(x)|}{\alpha_+(t)}\right) - \alpha_+(t)^2\, \hat{f}\left(\sigma \frac{|B_0(x')|}{\alpha_+(t)}\right)\right| &= \alpha_+(t)^2 |\hat{f}\,(\mathfrak{b}) - \hat{f}\,(\mathfrak{b}')\,| \\ &\leq C \left(\log \frac{1}{\rho}\right) \Big||B_0(x)| - |B_0(x')|\Big|\,. \end{aligned} \tag{4.6.15}$$

Therefore, using also the Lipschitz property for $x \mapsto |B_0(x)|$, we get that $\Omega_\rho \ni x \mapsto \phi(t,x)$ is uniformly Lipschitz for $t \in \mathcal{I}_+$.
Case 2 : $(\alpha_+(t) \leq \epsilon)$. We observe that for $(x,t) \in \Omega_\rho \times \mathcal{I}_-$,

$$\frac{\sigma\,|B_0(x)|}{\alpha_+(t)} \geq \frac{\sigma\,|B_0(x)|}{\epsilon}\,.$$

We note that $\hat{f}(\mathfrak{b}) = \frac{1}{2}$, $\forall \mathfrak{b} \geq 1$ (see [18, Theorem 2.1]). For this reason we choose

$$\epsilon = \frac{\lambda_{\min}}{2}\,\rho\,,$$

which implies that for $(x,t) \in \Omega_\rho \times \mathcal{I}_-$,

$$\frac{\sigma\,|B_0(x)|}{\alpha_+(t)} \geq 2 \qquad \text{and} \qquad \hat{f}\left(\sigma\frac{|B_0(x)|}{\alpha_+(t)}\right) = \frac{1}{2}\,.$$

Thus, for any $t \in \mathcal{I}_-$ and for any $x, x' \in \overline{\Omega}_\rho$, we get

$$\begin{aligned}\left|\alpha_+(t)^2\,\hat{f}\left(\sigma\frac{|B_0(x)|}{\alpha_+(t)}\right) - \alpha_+(t)^2\,\hat{f}\left(\sigma\frac{|B_0(x')|}{\alpha_+(t)}\right)\right| &= \left|\frac{\alpha_+(t)^2}{2} - \frac{\alpha_+(t)^2}{2}\right| \\ &= 0\,. \end{aligned} \tag{4.6.16}$$

Hence we get that $\Omega_\rho \ni x \mapsto \phi(t,x)$ is uniformly Lipschitz for $t \in \mathcal{I}_-$.

Now, we apply Lemma 4.6.4 with $D = \Omega_\rho$ and $M = \kappa^{\frac{1}{2}}$ and we obtain,

$$\int_{\Omega_\rho} a_+(x,\kappa)^2\,\hat{f}\left(\sigma\frac{|B_0(x)|}{a_+(x,\kappa)}\right)dx = \int_{\Omega_\rho}\overline{\phi}(x)\,dx + \mathcal{O}_\rho(\kappa^{-\frac{1}{2}})\,, \tag{4.6.17}$$

where $\overline{\phi}$ is introduced in (4.6.11).
Coming back to the integral over Ω, we get, for any $\rho \in (0,\rho_0)$ and for any $\kappa \geq \kappa_0$ with ρ_0 small enough and κ_0 large enough,

$$\int_{\Omega} a_+(x,\kappa)^2\,\hat{f}\left(\sigma\frac{|B_0(x)|}{a_+(x,\kappa)}\right)dx = \int_{\Omega}\overline{\phi}(x)\,dx + \mathcal{O}(\rho) + \mathcal{O}_\rho(\kappa^{-\frac{1}{2}})\,. \tag{4.6.18}$$

Here, we have used the fact that $\overline{\phi}$ is a bounded function in Ω. Let us show that the remainder term $s(\kappa)$ in the right hand side in (4.6.18) is $o(1)$. The remainder term has the form $s_1(\kappa)+s_2(\kappa)$ with $s_1(\kappa) = \mathcal{O}(\rho)$ and $s_2(\kappa) = \mathcal{O}_\rho(\kappa^{-\frac{1}{2}})$. Let us show that it is $o(1)$. Given $\varepsilon > 0$, there exists $\rho_\varepsilon > 0$ such that $|s_1(\kappa)| \leq \frac{\varepsilon}{2}$, for all $\kappa \geq \kappa_0$. Then, $\rho = \rho_\varepsilon$ being chosen, we can find $\kappa_\varepsilon \geq \kappa_0$ such that, for any $\kappa \geq \kappa_\epsilon$, $|s_2(\kappa)| \leq \frac{\varepsilon}{2}$.

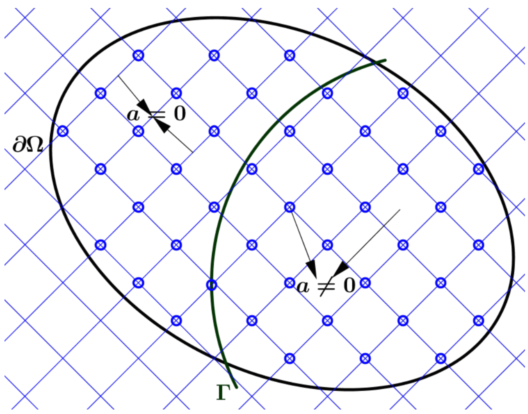

FIGURE 4.2 – Schematic representation of a domain with a κ-dependent oscillation pinning and with vanishing magnetic field along Γ.

$\square$

Proposition 4.6.7 (Verification of (A_4)). *Suppose that the function α defined in Proposition 4.6.5 satisfies*

$$|\alpha| + |\nabla\alpha| > 0 \quad \text{in } \mathbb{R}^2 . \tag{4.6.19}$$

Then Assumption (A_4) is satisfied.

Proof. Using (4.6.19), a change of variable $y = \kappa^{\frac{1}{2}}\, x$ and $\gamma' = \kappa^{\frac{1}{2}}\,\gamma$ yields,

$$\begin{aligned}\mathrm{card}\,\{\gamma \in \Gamma_\ell \cap \Omega \text{ with } Q_\ell(\gamma) \cap \partial\{x \in \Omega :\, a(x,\kappa) > 0\} \neq \emptyset\} \\ = \mathrm{card}\,\{\gamma' \in \Gamma_{\kappa^{\frac{1}{2}}\ell} \cap \kappa^{\frac{1}{2}}\Omega \text{ with } Q_{\kappa^{\frac{1}{2}}\ell}(\gamma') \cap \widehat{\Gamma} \neq \emptyset\}\,,\end{aligned}$$

where,

$$\widehat{\Gamma} = \{y \in \mathbb{R}^2 \,|\, \alpha(y) = 0\}\,.$$

Let $\epsilon \in (0,1)$, we introduce the domain

$$\widehat{D}_{\epsilon,M} = \{y \in M\ \cdot\ \Omega : \mathrm{dist}(y, \widehat{\Gamma}) \leq \epsilon\}\,.$$

Thanks to (4.6.8) and the periodicity assumption, we get the existence of positive constants C, M_0 and ϵ_0 such that, for any $\epsilon \in (0, \epsilon_0)$, $M \geq M_0$

$$|\widehat{D}_{\epsilon,M}| \leq C\, M\, \epsilon\,.$$

In the sequel, we choose $M = \kappa^{\frac{1}{2}}$ and $\epsilon = M\sqrt{2}\,\ell$. We note that, there exist constants $c > 0$ and $\kappa_0 > 0$ such that,

$$\forall \kappa \geq \kappa_0\,, \quad \forall \ell \leq c\,\kappa^{-\frac{1}{2}}\,, \qquad 0 < \epsilon \leq \epsilon_0\,.$$

We now observe that all the $Q_{\kappa^{\frac{1}{2}}\ell}(\gamma)$ touching $\widehat{\Gamma}$ are inside $\widehat{D}_{\kappa^{\frac{1}{2}}\sqrt{2}\,\ell, \kappa^{\frac{1}{2}}}$, hence we get, by comparison of the areas

$$\kappa\,\ell^2 \mathrm{card}\,\{\gamma' \in \Gamma_{\kappa^{\frac{1}{2}}\ell} \cap \kappa^{\frac{1}{2}}\Omega \text{ with } Q_{\kappa^{\frac{1}{2}}\ell}(\gamma') \cap \widehat{\Gamma}_\kappa \neq \emptyset\} \leq C\sqrt{2}\,\kappa\,\ell\,.$$

There exist positive constants C_1 and C_2, such that,

$$\forall \kappa \geq \kappa_0\,, \forall \ell \leq C_2 \kappa^{-\frac{1}{2}}\,, \mathrm{card}\,\{\gamma \in \Gamma_\ell \cap \Omega \text{ with } Q_\ell(\gamma) \cap \partial\{x \in \Omega :\, a(x,\kappa) > 0\} \neq \emptyset\} \leq C_1\,\ell^{-1}\,.$$

□

4.6.1.3 Second example.

This example was considered by Aftalion, Sandier and Serfaty (see (H_2)).

Proposition 4.6.8. *Suppose that* (4.1.4) *and* (4.1.15) *hold. Let $a(x,\kappa) = a(x) + \beta(x,\kappa)$, where $\beta(x,\kappa)$ is a nonnegative function and $\{a > 0\} \cap \Omega \neq \varnothing$, (see Fig. 4.3). There exist positive constants τ_1 and κ_0 such that,*

$$\forall \kappa \geq \kappa_0\,, \qquad E_{\mathrm{g}}^{\boldsymbol{L}}(\kappa, H, a, B_0) \geq \tau_1\,\kappa^2\,.$$

Proof. We can write,

$$\kappa^2 \int_{\{a(x,\kappa)>0\}} a(x,\kappa)^2 \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) dx \geq \kappa^2 \int_{\{a(x)>0\}} a(x,\kappa)^2 \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) dx$$
$$\geq \kappa^2 \int_{\{a(x)>0\}} a(x)^2 \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{\overline{a}}\right) dx\,. \qquad (4.6.20)$$

Here we have used that $\hat{f}$ is increasing, the nonnegativity of β to get $a(x,\kappa) \geq a(x)$, Assumption (A_2) to estimate $\hat{f}$ from below, and $\{a(x)>0\} \subset \{a(x,\kappa)>0\}$.
Proceding like in (4.6.4), there exist $\tau_1 > 0$ and $\kappa_0 > 0$ such that,

$$\forall \kappa \geq \kappa_0\,, \qquad \kappa^2 \int_{\{a(x,\kappa)>0\}} a(x,\kappa)^2 \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) dx \geq \tau_1 \kappa^2\,. \qquad (4.6.21)$$

□

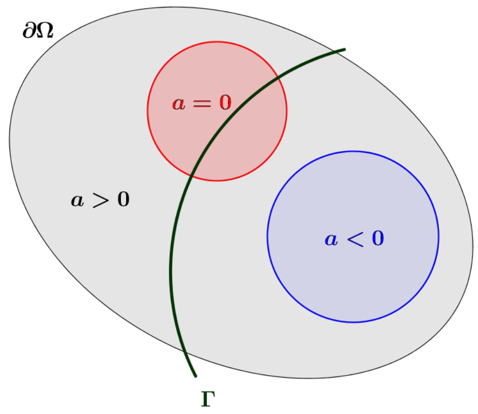

FIGURE 4.3 – Schematic representation of some domain with pinning term dependent of κ and with vanishing magnetic field along Γ.

4.6.1.4 Third example :

This example is similar to the previous example, but here we suppose that

$$\beta(x,\kappa) = \alpha(\kappa^{\frac{1}{2}}x)\,,$$

where $\alpha(\cdot)$ is a Γ_{T_1,T_2}-periodic positive function in $\mathbb{R}^2$.

Proposition 4.6.9. *Suppose that* (4.1.4) *and* (4.1.15) *hold. Let* $a(x,\kappa) = a(x) + \alpha(\kappa^{\frac{1}{2}}x)$, *where* $\alpha(\cdot)$ *is a* Γ_{T_1,T_2}*-periodic positive bounded function in* $\mathbb{R}^2$, $a(\cdot) \in C^1(\overline{\Omega})$ *and* $\{a<0\} \cap \Omega = \varnothing$. *Then the leading order term* $E_{\rm g}^{\boldsymbol{L}}$ *defined in* (4.6.1) *satisfies,*

$$E_{\rm g}^{\boldsymbol{L}}(\kappa, H, a, B_0) = \kappa^2 \int_\Omega \overline{\phi}(x)\, dx + o(\kappa^2)\,, \quad \text{as } \kappa \to +\infty\,.$$

Here,

$$\overline{\phi}(x) = \frac{1}{T_1T_2}\int_0^{T_1}\int_0^{T_2}(a(x)+\alpha(t_1,t_2))^2\,\hat{f}\left(\sigma\frac{|B_0(x)|}{a(x)+\alpha(t_1,t_2)}\right)\,dt_1dt_2\,.$$

The proof of Proposition 4.6.9 is similar to that of Proposition 4.6.5.

4.6.2 Upper bound of the main term.

It is easy to show that $E_{\mathrm{g}}^{\mathbf{L}}$ is less than $C\kappa^2$ for some $C>0$. Indeed, using the bound of a in (4.1.6) and the bound $\hat{f}(b)\leq\frac{1}{2}$, we have,

$$\kappa^2\int_{\{a(x,\kappa)>0\}}a(x,\kappa)^2\,\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)\,dx\leq C\kappa^2\,,$$

and

$$\frac{\kappa^2}{2}\int_{\{a(x,\kappa)\leq 0\}}a(x,\kappa)^2\,dx\leq C\kappa^2\,.$$

4.7 Proof of Theorem 4.1.5

The technique that will be used in this proof has been introduced by Helffer-Kachmar in [27] for the case $a(x,\kappa)=1$. The proof is decomposed into three steps :

Step 1 : Case $\mathcal{D}=\Omega$.

Let $(\psi,\mathbf{A})$ be a solution of (4.1.12). Thanks to (4.4.3), we have,

$$\begin{aligned}\int_\Omega|(\nabla-i\kappa H\mathbf{A})\psi|^2\,dx &= \kappa^2\int_\Omega(a(x,\kappa)-|\psi|^2)|\psi|^2\,dx\\ &= \frac{\kappa^2}{2}\int_\Omega(a(x,\kappa)^2-|\psi(x)|^4)\,dx-\frac{\kappa^2}{2}\int_\Omega(a(x,\kappa)-|\psi|^2)^2\,dx\,.\end{aligned}$$

Having in mind the definition of $\mathcal{E}_0(\psi,\mathbf{A};a,\Omega)$, we get,

$$\frac{\kappa^2}{2}\int_\Omega(a(x,\kappa)^2-|\psi(x)|^4)\,dx=\mathcal{E}_0(\psi,\mathbf{A};a,\Omega)\,. \tag{4.7.1}$$

Using (4.5.24), we get that as $\kappa\longrightarrow+\infty$

$$\begin{aligned}\frac{\kappa^2}{2}\int_\Omega(a(x,\kappa)^2-|\psi(x)|^4)\,dx &= \kappa^2\int_{\{a(x,\kappa)>0\}}a(x,\kappa)^2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)\,dx\\ &\quad+\frac{\kappa^2}{2}\int_{\{a(x,\kappa)\leq 0\}}a(x,\kappa)^2\,dx+o\left(\kappa^2\right)\,.\end{aligned} \tag{4.7.2}$$

Notice that

$$\int_\Omega a(x,\kappa)^2\,dx=\int_{\{a(x,\kappa)\leq 0\}}a(x,\kappa)^2\,dx+\int_{\{a(x,\kappa)>0\}}a(x,\kappa)^2\,dx\,.$$

Therefore, dividing (4.7.2) by κ^2, we get

$$\int_\Omega |\psi(x)|^4\,dx = -\int_{\{a(x,\kappa)>0\}} a(x,\kappa)^2 \left\{2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)\,dx - 1\right\}\,dx + o\,(1)\,. \tag{4.7.3}$$

Step 2 : Upper bound.
Let $\mathcal{D}\subset\Omega$ be a regular domain and, for $\ell\in(0,1)$,

$$\mathcal{D}_\ell = \{x\in\mathcal{D} : \mathrm{dist}(x,\partial\mathcal{D})\geq \ell\}\,. \tag{4.7.4}$$

We introduce a cut-off function $\chi_\ell\in C_c^\infty(\mathbb{R}^2)$ such that

$$0\leq\chi_\ell\leq 1 \text{ in } \mathbb{R}^2\,,\quad \mathrm{supp}\chi_\ell\subset\mathcal{D}\,,\quad \chi_\ell=1 \text{ in } \mathcal{D}_\ell \quad\text{and}\quad |\nabla\chi_\ell|\leq\frac{C}{\ell} \text{ in } \mathbb{R}^2\,, \tag{4.7.5}$$

where C is a positive constant. We multiply both sides of $(4.1.12)_a$ by $\chi_\ell^2\psi$. It results from an integration by parts that

$$\begin{aligned}\int_\mathcal{D}\left(|(\nabla - i\kappa H\mathbf{A})\chi_\ell\psi|^2 - \kappa^2 a\,\chi_\ell^2|\psi|^2+\kappa^2\chi_\ell^2|\psi|^4\right)\,dx &= \int_\mathcal{D}|\nabla\chi_\ell|^2\,|\psi|^2\,dx\\ &= \mathcal{O}(\ell^{-1})\,.\end{aligned} \tag{4.7.6}$$

Here, we have used the fact that $|\nabla\,\chi_\ell|^2=\mathcal{O}(\ell^{-2})$, $|\mathcal{D}_\ell|=\mathcal{O}(\ell)$ and the bound of ψ in (4.4.9). We notice that $\chi_\ell^4\leq\chi_\ell^2\leq 1$. We add to both sides the term $\frac{\kappa^2}{2}\int_\mathcal{D} a^2\,dx$ to obtain,

$$\int_\mathcal{D}\left(|(\nabla - i\kappa H\mathbf{A})\chi_\ell\psi|^2+\frac{\kappa^2}{2}a^2-\kappa^2 a\,|\chi_\ell\,\psi|^2+\kappa^2|\chi_\ell\,\psi|^4\right)\,dx\leq\,C\,\ell^{-1}+\frac{\kappa^2}{2}\int_\mathcal{D} a^2\,dx\,.$$

This implies that

$$\mathcal{E}_0(\chi_\ell\psi,\mathbf{A};a,\mathcal{D})\leq\frac{\kappa^2}{2}\int_\mathcal{D}(a^2-\chi_\ell^4|\psi|^4)\,dx+C\,\ell^{-1}\,.$$

Using (4.7.5), we get

$$\begin{aligned}\int_\mathcal{D}|\psi|^4\,dx &= \int_\mathcal{D}\chi_\ell^4|\psi|^4\,dx+\int_\mathcal{D}(1-\chi_\ell^4)|\psi|^4\,dx\\ &\leq\int_\mathcal{D}\chi_\ell^4|\psi|^4\,dx+C'\,\ell\,,\end{aligned} \tag{4.7.7}$$

and consequently,

$$\mathcal{E}_0(\chi_\ell\psi,\mathbf{A};a,\mathcal{D})\leq\frac{\kappa^2}{2}\int_\mathcal{D}(a^2-|\psi|^4)\,dx+C(\ell^{-1}+\ell)\,. \tag{4.7.8}$$

Using (4.5.22) with $h=\chi_\ell$ and taking the choice of ℓ defined in (4.3.18), we get, as $\kappa\to+\infty$,

$$\begin{aligned}\frac{\kappa^2}{2}\int_\mathcal{D}(a^2-|\psi|^4)\,dx\geq\kappa^2\int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)\,dx+\frac{\kappa^2}{2}\int_{\mathcal{D}\cap\{a(x,\kappa)\leq 0\}} a(x,\kappa)^2\,dx\\ +\,o\left(\kappa^2\right)\,.\end{aligned} \tag{4.7.9}$$

Notice that,

$$\int_{\mathcal{D}} a(x,\kappa)^2\,dx = \int_{\mathcal{D}\cap\{a(x,\kappa)\le 0\}} a(x,\kappa)^2\,dx + \int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2\,dx\,.$$

Therefore,

$$-\frac{\kappa^2}{2}\int_{\mathcal{D}} |\psi|^4\,dx \ge \kappa^2 \int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2 \hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right)dx - \frac{\kappa^2}{2}\int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2\,dx + o\left(\kappa^2\right)\,. \quad (4.7.10)$$

Dividing both sides by $-\frac{\kappa^2}{2}$, we obtain, as $\kappa \longrightarrow +\infty$,

$$\int_{\mathcal{D}} |\psi|^4\,dx \le -\int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2 \left\{2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) - 1\right\}dx + o\,(1)\,. \quad (4.7.11)$$

Remark 4.7.1. We can replace $\mathcal{D}$ by $\overline{\mathcal{D}}^c$ such that the estimate in (4.7.11) is still true. That is :

$$\int_{\overline{\mathcal{D}}^c} |\psi|^4\,dx \le -\int_{\overline{\mathcal{D}}^c\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2 \left\{2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) - 1\right\}dx + o\,(1)\,. \quad (4.7.12)$$

Step 3 : Lower bound.

We can decompose $\int_{\mathcal{D}} |\psi|^4\,dx$ as follows :

$$\int_{\mathcal{D}} |\psi|^4\,dx = \int_{\Omega} |\psi|^4\,dx - \int_{\overline{\mathcal{D}}^c} |\psi|^4\,dx$$

Thanks to Remark 4.7.1, using the asymptotics in (4.7.3) obtained in Step 1 when $\mathcal{D} = \Omega$ and the upper bound in Step 2 , we get

$$\int_{\mathcal{D}} |\psi|^4\,dx \le -\int_{\mathcal{D}\cap\{a(x,\kappa)>0\}} a(x,\kappa)^2 \left\{2\hat{f}\left(\frac{H}{\kappa}\frac{|B_0(x)|}{a(x,\kappa)}\right) - 1\right\}dx + o\,(1)\,. \quad (4.7.13)$$

4.8 Extension of the Giorgi-Phillips Theorem

In this section we extend a result of Giorgi-Phillips [23], in the two cases when the external magnetic field B_0 is variable (i.e. $\Gamma \neq \varnothing$) and when the external magnetic field B_0 is constant (i.e. $\Gamma = \varnothing$), with a pinning term. We recall that the normal solution $(0, \mathbf{F})$ is a trivial solution of the Ginzburg-Landau system (4.1.12). We will show that this solution is a global minimizer, when κ and H are sufficiently large. We first establish a priori estimates for a critical point $(\psi, \mathbf{A})$ of the G-L-functional.

4.8.1 Estimates of A and of $\|(\nabla - i\kappa H\mathbf{F})\psi\|$.

We need the following estimates on $\mathbf{A}$ and on $\|(\nabla - i\kappa H\mathbf{F})\psi\|$ which are independent of the assumption of Γ.

Theorem 4.8.1. *There exist positive constants C_1, C_2 and C_3 such that, if* (4.1.6) *hold,* $\kappa > 0$, $H > 0$ *and* $(\psi, \mathbf{A})$ *is a solution of* (4.1.12), *then,*

$$\|(\nabla - i\kappa H\mathbf{A})\psi\|_{L^2(\Omega)} \leq C_1\,\kappa\,\|\psi\|_{L^2(\Omega)}\,, \tag{4.8.1}$$

$$\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)} \leq \frac{C_2}{H}\,\|\psi\|_{L^2(\Omega)}\|\psi\|_{L^4(\Omega)}\,, \tag{4.8.2}$$

$$\|(\nabla - i\kappa H\mathbf{F})\psi\|_{L^2(\Omega)} \leq C_3\,\kappa\,\|\psi\|_{L^2(\Omega)}\,. \tag{4.8.3}$$

Proof. **We first prove** (4.8.1)**.** In the case when $\overline{a} \leq 0$ with $\overline{a}$ introduced in (4.1.10), we get using (4.4.9) that $\psi = 0$ and hence (4.8.1) is proved.
In the case when $\overline{a} > 0$, thanks to (4.4.9), we have,

$$0 \leq (\overline{a} - |\psi|^2) \leq \overline{a}\,. \tag{4.8.4}$$

We recall that if $(\psi, \mathbf{A})$ is a solution of (4.1.12) then, (see (4.4.3))

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A})\psi|^2\,dx = \kappa^2 \int_\Omega (a(x,\kappa) - |\psi|^2)|\psi|^2\,dx\,.$$

Using (4.1.6) and (4.8.4), we obtain (4.8.1).
Now, we prove (4.8.2)**.** We obtain from the equation in $(4.1.12)_b$ the following estimate (see [14, Equation (11.9b)]) :

$$\kappa H \int_\Omega |\operatorname{curl}(\mathbf{A}-\mathbf{F})|^2\,dx \leq \|(\nabla - i\kappa H\mathbf{A})\psi\|_{L^2(\Omega)}\,\|(\mathbf{A}-\mathbf{F})\psi\|_{L^2(\Omega)}\,.$$

Using (4.8.1) and applying Hölder's inequality, we get

$$\kappa H \int_\Omega |\operatorname{curl}(\mathbf{A}-\mathbf{F})|^2\,dx \leq C\,\kappa\,\|\psi\|_{L^2(\Omega)}\|\psi\|_{L^4(\Omega)}\,\|\mathbf{A}-\mathbf{F}\|_{L^4(\Omega)}\,.$$

We get by regularity of the curl-div system (see [14, A.7]),

$$\|\mathbf{A}-\mathbf{F}\|_{H^1(\Omega)} \leq C\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)}\,, \tag{4.8.5}$$

where C is a positive constant.
By the Sobolev embedding theorem, we get,

$$\begin{aligned}\|\mathbf{A}-\mathbf{F}\|_{L^4(\Omega)} &\leq C_{\text{Sob}}\,\|\mathbf{A}-\mathbf{F}\|_{H^1(\Omega)}\\ &\leq \widehat{C}\,\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)}\,.\end{aligned} \tag{4.8.6}$$

Consequently,

$$\kappa H \int_\Omega |\operatorname{curl}(\mathbf{A}-\mathbf{F})|^2\,dx \leq \kappa\,\|\psi\|_{L^2(\Omega)}\|\psi\|_{L^4(\Omega)}\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)}\,,$$

which leads to (4.8.2).

Finally, we prove (4.8.3)**.** Using (4.8.2) and (4.8.6), Hölder's inequality gives,

$$\begin{aligned}\|(\mathbf{A}-\mathbf{F})\psi\|^2_{L^2(\Omega)} &\leq \|\mathbf{A}-\mathbf{F}\|^2_{L^4(\Omega)}\|\psi\|^2_{L^4(\Omega)}\\ &\leq \frac{C'}{H^2}\|\psi\|^4_{L^4(\Omega)}\|\psi\|^2_{L^2(\Omega)}\,,\end{aligned} \tag{4.8.7}$$

Using (4.8.1), (4.8.7) and the bound of ψ above, Young's inequality gives,

$$\begin{aligned}\|(\nabla - i\kappa H\mathbf{F})\psi\|^2_{L^2(\Omega)} &\leq 2\|(\nabla - i\kappa H\mathbf{A})\psi\|^2_{L^2(\Omega)} + 2\,(\kappa H)^2\|(\mathbf{A}-\mathbf{F})\psi\|^2_{L^2(\Omega)}\\ &\leq 2\,C''\,\kappa^2\|\psi\|^2_{L^2(\Omega)}\,.\end{aligned} \tag{4.8.8}$$

□

4.8.2 The case $\Gamma = \varnothing$.

For $\xi \in \mathbb{R}$, we consider the Neumann realization $\mathfrak{h}^{N,\xi}$ in $L^2(\mathbb{R}_+)$ associated with the operator $-\frac{d^2}{dt^2} + (t+\xi)^2$, i.e.

$$\mathfrak{h}^{N,\xi} := -\frac{d^2}{dt^2} + (t+\xi)^2\,, \qquad \mathcal{D}(\mathfrak{h}^{N,\xi}) = \{u \in B^2(\mathbb{R}_+) : u'(0) = 0\}\,, \tag{4.8.9}$$

where,

$$B^2(\mathbb{R}_+) = \{u \in L^2(\mathbb{R}_+) : t^p u^{(q)} \in L^2(\mathbb{R}_+), \forall p,q \in \mathbb{N}\ s.t.\ p+q \leq 2\}\,.$$

M. Dauge and B. Helffer [12] (see also Fournais-Helffer [14, Proposition 4.2.2]) have proved that the lowest eigenvalue μ of $\mathfrak{h}^{N,\xi}$ admits a minimum Θ_0, which is attained at a unique point $\xi_0 < 0$, and satisfies :

$$\Theta_0 = \inf_{\xi} \mu(\xi) = \mu(\xi_0) < 1\,. \tag{4.8.10}$$

Moreover

$$\Theta_0 = \xi_0^2\,. \tag{4.8.11}$$

We introduce the notation :

$$\inf_{x\in\overline{\Omega}} |B_0(x)| = b_0 \qquad \text{and} \qquad \inf_{x\in\partial\Omega} |B_0(x)| = b_0'\,. \tag{4.8.12}$$

We denote by $\mu^N(\mathcal{B}\mathbf{F};\Omega)$ the lowest eigenvalue of the Schrödinger operator $P^{\Omega}_{\mathcal{B}\mathbf{F},0}$ (see (4.1.13)) with Neumann condition in $L^2(\Omega)$:

$$\mu^N(\mathcal{B}\mathbf{F};\Omega) = \inf_{\substack{\psi\in H^1(\Omega)\\ \psi\neq 0}} \frac{\langle P^{\Omega}_{\mathcal{B}\mathbf{F},0}\,\psi,\psi\rangle}{\|\psi\|^2_{L^2(\Omega)}}\,. \tag{4.8.13}$$

In [14], it is proved that

Theorem 4.8.2. *Suppose that $\Omega \subset \mathbb{R}^2$ is an open bounded set with smooth boundary and $\Gamma = \varnothing$.*

Then,

$$\lim_{\mathcal{B}\longrightarrow+\infty}\frac{\mu^N(\mathcal{B}\mathbf{F},\Omega)}{\mathcal{B}}=\min(b_0,\Theta_0\, b_0')\,. \tag{4.8.14}$$

In the next theorem, we give a simple proof of the result which says that $(0,\mathbf{F})$ is the unique minimizer of the functional when H is sufficiently large and when the magnetic field B_0 is constant with pinning term.

Theorem 4.8.3. *Let $\Omega\subset\mathbb{R}^2$ be a smooth, bounded, simply-connected open set and $\Gamma=\varnothing$. Then, there exist positive constants C and κ_0, such that, if*

$$H\geq C\kappa\,,\qquad \kappa\geq\kappa_0\,,$$

then $(0,\mathbf{F})$ is the unique solution to (4.1.12).

Proof. We assume that we have a **non normal** critical point $(\psi,\mathbf{A})$ for $\mathcal{E}_{\kappa,H,a,B_0}$. This means that $(\psi,\mathbf{A})\in H^1(\Omega)\times H^1_{\mathrm{div}}(\Omega)$ is a solution of (4.1.12) and

$$\int_\Omega|\psi|^2\,dx>0\,. \tag{4.8.15}$$

Therefore, we get from (4.4.9) that,

$$|\psi(x)|^2\leq\overline{a}\qquad\forall x\in\overline{\Omega}\,,$$

where $\overline{a}$ is introduced in (4.1.10).
Let

$$\mathcal{B}=\kappa H\,. \tag{4.8.16}$$

Theorem 4.8.1 tells us that,

$$\|(\nabla-i\mathcal{B}\mathbf{F})\psi\|^2_{L^2(\Omega)}\leq C\,\kappa^2\,\|\psi\|^2_{L^2(\Omega)}\,.$$

Since ψ satisfies (4.8.15), this implies by assumption that the lowest Neumann eigenvalue $\mu^N(\mathcal{B}\mathbf{F};\Omega)$ of $P^\Omega_{\mathcal{B}\mathbf{F},0}$ in $L^2(\Omega)$ satisfies,

$$\mu^N(\mathcal{B}\mathbf{F};\Omega)\leq C\,\kappa^2\,. \tag{4.8.17}$$

Thanks to Theorem 4.8.2, we get the existence of a constant $C>0$, such that, if $H\geq C\,\kappa$, then $(0,\mathbf{F})$ is the unique solution to (4.1.12). □

4.8.3 The case $\Gamma\neq\varnothing$.

We recall the definition of λ_0 in (4.1.31), the definition of Γ in (4.1.3) and for any $\theta\in(0,\pi)$ we recall that $\lambda(\mathbb{R}^2_+,\theta)$ is the bottom of the spectrum of the operator $P^{\mathbb{R}^2_+}_{\mathbf{A}_{\mathrm{app},\theta},0}$, with

$$\mathbf{A}_{\mathrm{app},\theta} = -\left(\frac{x_2^2}{2}\cos\theta, \frac{x_1^2}{2}\sin\theta\right).$$

Define

$$\alpha_1(B_0) = \min\left\{\lambda_0^{\frac{3}{2}}\min_{x\in\Gamma\cap\Omega}|\nabla B_0(x)|, \min_{x\in\Gamma\cap\partial\Omega}\lambda(\mathbb{R}^2_+,\theta(x))^{\frac{3}{2}}|\nabla B_0(x)|\right\}. \tag{4.8.18}$$

In [42], it is proved that

Theorem 4.8.4. *Suppose that* (4.1.4) *holds and* $\Gamma \neq \varnothing$*. Then*

$$\lim_{\mathcal{B}\longrightarrow+\infty}\frac{\mu^N(\mathcal{B}\mathbf{F},\Omega)}{\mathcal{B}^{\frac{2}{3}}} = \alpha_1(B_0)^{\frac{2}{3}}. \tag{4.8.19}$$

In the next theorem, we give a simple proof of the result which says that $(0,\mathbf{F})$ is the unique minimizer of the functional when H is sufficiently large and when B_0 is variable. This result was obtained in [23] for the case with constant magnetic field and with a constant pinning term.

Theorem 4.8.5. *Let* $\Omega\subset\mathbb{R}^2$ *be a smooth, bounded, simply-connected open set, the pinning term* a *satisfying* (4.1.6)*, and the magnetic field* B_0 *satisfying* (4.1.4)*. Then, there exist positive constants* C *and* κ_0*, such that, if*

$$H \geq C\kappa^2, \qquad \kappa\geq\kappa_0.$$

Then $(0,\mathbf{F})$ *is the unique solution to* (4.1.12)*.*

Proof. Similarly to the proof of Theorem 4.8.3, we assume that we have a **non normal** critical point $(\psi,\mathbf{A})$ for $\mathcal{E}_{\kappa,H,a,B_0}$.
Therefore, we get from (4.8.3) that,

$$\mu^N(\mathcal{B}\mathbf{F};\Omega)\leq C\,\kappa^2 \quad (\mathcal{B}=\kappa H).$$

Thanks to Theorem 4.8.4, we get a contradiction, if $H\geq C\kappa^2$ and C is sufficiently large. □

4.9 Asymptotics of $\mu_1(\kappa, H)$: the case with non vanishing magnetic field

The aim of this section is to give an estimate for the lowest eigenvalue $\mu_1(\kappa,H)$ of the operator $P^{\Omega}_{\kappa H\mathbf{F},-\kappa^2 a}$ (see (4.1.26)) in the case when $\Gamma=\varnothing$ with a κ-independent pinning (i.e. $a(x,\kappa)=a(x)$). Recall that the set Γ is introduced in (4.1.3).

Without loss of generality we suppose that $B_0>0$ in $\overline{\Omega}$. Our results will be formulated by introducing :

$$\Lambda_1(B_0,a,\sigma)=\min\left\{\inf_{x\in\Omega}\{\sigma\, B_0(x)-a(x)\}\,,\,\inf_{x\in\partial\Omega}\{\Theta_0\,\sigma\, B_0(x)-a(x)\}\right\}\,, \tag{4.9.1}$$

where σ is a positive constant.
In the case when the pinning term is constant (i.e. $a(x)=a_0$), (4.9.1) becomes as follows :

$$\Lambda_1(B_0,a,\sigma)=\sigma\min\left\{\inf_{x\in\Omega}\{B_0(x)\}\,,\Theta_0\inf_{x\in\partial\Omega}\{B_0(x)\}\right\}-a_0\,.$$

This case was treated by Pan and Kwek [34].
Let $\mathcal{Q}^{\Omega}_{\mathcal{B}\mathbf{F},-\frac{\mathcal{B}}{\sigma}a}$ be the quadratic form of $P^{\Omega}_{\mathcal{B}\mathbf{F},-\frac{\mathcal{B}}{\sigma}a}$, i.e.

$$\mathcal{Q}^{\Omega}_{\mathcal{B}\mathbf{F},-\frac{\mathcal{B}}{\sigma}a}(\psi)=\int_\Omega\left(|(\nabla-i\mathcal{B}\mathbf{F})\psi|^2-\frac{\mathcal{B}}{\sigma}\,a(x)|\psi|^2\right)\,dx\,. \tag{4.9.2}$$

Proposition 4.9.1. *Let $\Omega\subset\mathbb{R}^2$ be an open bounded set with smooth boundary, I a closed interval in $(0,+\infty)$ and $\Gamma=\varnothing$. There exist positive constant C and $\mathcal{B}_0$ such that if $\sigma\in I$, $\mathcal{B}\geq\mathcal{B}_0$, $\psi\in H^1(\Omega)\setminus\{0\}$ and $a\in C^1(\overline{\Omega})$, then,*

$$\frac{\mathcal{Q}^{\Omega}_{\mathcal{B}\mathbf{F},-\frac{\mathcal{B}}{\sigma}a}(\psi)}{\|\psi\|^2_{L^2(\Omega)}}\geq\frac{\mathcal{B}}{\sigma}\,\Lambda_1(B_0,a,\sigma)-C\,\mathcal{B}^{\frac{3}{4}}\,. \tag{4.9.3}$$

Proof. The proof is a consequence of the following inequality that we take from [14, Prop. 9.2.1],

$$\forall\,\psi\in H^1(\Omega)\,,\quad\int_\Omega|(\nabla-i\mathcal{B}\mathbf{F})\psi|^2\,dx\geq\int_\Omega\left(U(x)-\bar{C}B^{3/4}\right)|\psi|^2\,dx\,,$$

where

$$U(x)=\begin{cases}\mathcal{B}\,B_0(x) & \text{if } \mathrm{dist}(x,\partial\Omega)\geq B^{-3/8}\,,\\ \Theta_0\mathcal{B}\,B_0(x) & \text{if } \mathrm{dist}(x,\partial\Omega)< B^{-3/8}\,,\end{cases} \tag{4.9.4}$$

$\mathcal{B}\geq\bar{\mathcal{B}}_0$, $\bar{\mathcal{B}}_0$ and $\bar{C}$ are two constants independent of $\mathcal{B}$.

Clearly, there exist two constants $C'>0$ and $\mathcal{B}_0>0$ such that, for all $\sigma\in I$, we have,

$$U(x)-\frac{\mathcal{B}}{\sigma}a(x)\geq\frac{\mathcal{B}}{\sigma}\Lambda_1(B_0,a,\sigma)-C'B^{3/4}\,.$$

□

Coming back to our initial parameters κ and H, we obtain :

Theorem 4.9.2. *Let $\Omega\subset\mathbb{R}^2$ be an open bounded set with smooth boundary and $\Gamma=\varnothing$. Suppose that (4.1.15) holds and $a\in C^1(\overline{\Omega})$, then,*

$$\mu_1(\kappa,H)\geq\kappa^2\,\Lambda_1\left(B_0,a,\frac{H}{\kappa}\right)+\mathcal{O}(\kappa^{\frac{3}{2}})\,,\qquad\text{as}\,\kappa\to+\infty\,.$$

Here, Λ_1 is introduced in (4.9.1).

Proof. We apply Proposition 4.9.1 with

$$\mathcal{B} = \kappa H\,, \quad \sigma = \frac{H}{\kappa} \quad \text{and} \quad I = [\lambda_{\min}, \lambda_{\max}]\,.$$

Let us verify that the conditions of the proposition are satisfied for this choice.
It is trivial that $\sigma \in I$. Now, as $\kappa \to +\infty$, we have,

$$\mathcal{B} = \sigma\,\kappa^2 \to +\infty\,.$$

This implies that, as $\kappa \to +\infty$,

$$\mu_1(\kappa, H) \geq \kappa^2\,\Lambda_1\left(B_0, a, \frac{H}{\kappa}\right) + \mathcal{O}(\kappa^{\frac{3}{2}})\,.$$

This finishes the proof of the theorem. □

4.9.1 Upper bound

Proposition 4.9.3 (Upper bound in the bulk)**.** *Suppose that $\Omega \subset \mathbb{R}^2$ is an open bounded set with smooth boundary $\partial\Omega$, $\lambda_{\max} > 0$ and $\Gamma = \varnothing$. For any $x_0 \in \Omega$, there exist positive constants C and $\mathcal{B}_0$ such that, if $\sigma \in (0, \lambda_{\max}]$, $\mathcal{B} \geq \mathcal{B}_0$ and $a \in C^1(\overline{\Omega})$, then,*

$$\mu_{\mathcal{B},\sigma} \leq \frac{\mathcal{B}}{\sigma}\,\{\sigma\,B_0(x_0) - a(x_0)\} + C\,\mathcal{B}^{\frac{1}{2}}\,. \tag{4.9.5}$$

Here,

$$\mu_{\mathcal{B},\sigma} = \inf_{\psi \in H^1(\Omega)\setminus\{0\}} \frac{\mathcal{Q}^{\Omega}_{\mathcal{B}\mathbf{F},-\frac{\mathcal{B}}{\sigma}a}(\psi)}{\|\psi\|^2_{L^2(\Omega)}}\,, \tag{4.9.6}$$

where $\mathcal{Q}^{\Omega}_{\mathcal{B}\mathbf{F},-\frac{\mathcal{B}}{\sigma}a}$ is introduced in (4.9.2)*.*

Proof. Thanks to (4.9.2), we have,

$$\mathcal{Q}^{\Omega}_{\mathcal{B}\mathbf{F},-\frac{\mathcal{B}}{\sigma}a}(u) = \int_{\Omega} |(\nabla - i\mathcal{B}\mathbf{F})u(x)|^2\,dx - \frac{\mathcal{B}}{\sigma}\int_{\Omega} a(x)|u(x)|^2\,dx\,.$$

The upper bound of the first term in the right hand side above is based on the construction of Gaussian quasi-mode (see [14, Subsection 2.4.2] for the case with constant pinning) centered at $x_0 \in \Omega$,

$$\varphi_1(x) = e^{i\,\mathcal{B}\,\phi_0}\,\chi\left(\mathcal{B}^{+\frac{1}{2}}(x - x_0)\right)\,u\left(\sqrt{\mathcal{B}B_0(x_0)}\,(x - x_0)\right)\,.$$

Here, χ is a cut-off function equal to 1 in a neighborhood of 0 such that $\operatorname{supp}\chi \subset D(0,1)$, the function ϕ_0 satisfies (4.3.4) and the function u defined as follows :

$$u(x) = \frac{\pi^{-\frac{1}{4}}}{\sqrt{2}}e^{-\frac{|x|^2}{2}}\,.$$

We note that $\operatorname{supp}\varphi_1 \subset \Omega$ for $\mathcal{B}$ large enough. As in [14, (2.35)], we get the existence of a positive constant $\mathcal{B}_0$ such that, for any $\mathcal{B} \geq \mathcal{B}_0$,

$$\frac{\int_\Omega |(\nabla - i\mathcal{B}\mathbf{F})\varphi_1(x)|^2\,dx}{\int_\Omega |\varphi_1(x)|^2\,dx} \leq \mathcal{B}\,B_0(x_0) + \mathcal{O}(\mathcal{B}^{\frac{1}{2}})\,. \tag{4.9.7}$$

To derive the upper bound for the second term, we use Taylor's formula for the function a near x_0,

$$|a(x) - a(x_0)| \leq C\,\mathcal{B}^{-\frac{1}{2}}\,, \qquad \left(x \in D\left(x_0, \mathcal{B}^{-\frac{1}{2}}\right)\right)\,. \tag{4.9.8}$$

Using (4.9.8), since $\operatorname{supp}\varphi_1 \subset D\left(x_0, \mathcal{B}^{-\frac{1}{2}}\right)$, we get,

$$-\int_\Omega a(x)|\varphi_1(x)|^2\,dx \leq -a(x_0)\int_\Omega |\varphi_1(x)|^2\,dx + C\,\mathcal{B}^{-\frac{1}{2}}\int_\Omega |\varphi_1(x)|^2\,dx\,, \tag{4.9.9}$$

and consequently

$$-\frac{\mathcal{B}}{\sigma}\frac{\int_\Omega a(x)|\varphi_1(x)|^2\,dx}{\int_\Omega |\varphi_1(x)|^2\,dx} \leq -\frac{\mathcal{B}}{\sigma}\,a(x_0) + C\,\mathcal{B}^{\frac{1}{2}}\,. \tag{4.9.10}$$

Collecting (4.9.7) and (4.9.10), we finish the proof of Proposition 4.9.6. □

Remark 4.9.4. When

$$\inf_{x\in\Omega}\{\sigma\,B_0(x) - a(x)\} < \inf_{x\in\partial\Omega}\{\Theta_0\,\sigma\,B_0(x) - a(x)\}\,,$$

we notice that, if the infimum of $\sigma\,B_0(x) - a(x)$ was attained on $\partial\Omega$, (i.e. there exists $x_0 \in \partial\Omega$ such that $\inf_{x\in\Omega}\{\sigma\,B_0(x) - a(x)\} = \sigma\,B_0(x_0) - a(x_0)$), we would have,

$$\sigma\,B_0(x_0) - a(x_0) < \Theta_0\,\sigma\,B_0(x_0) - a(x_0)\,,$$

which is impossible, since $\Theta_0 < 1$. Hence, we can choose $x_0 \in \Omega$, such that,

$$\sigma\,B_0(x_0) - a(x_0) = \inf_{x\in\Omega}\{\sigma\,B_0(x) - a(x)\}\,,$$

and we apply Proposition 4.9.3 with

$$\mathcal{B} = \kappa H \quad \text{and} \quad \sigma = \frac{H}{\kappa}\,.$$

Thus, we get the existence of a positive constant κ_0 such that, if,

$$\kappa \geq \kappa_0 \quad \text{and} \quad \kappa_0\,\kappa^{-1} < H < \lambda_{\max}\,\kappa\,, \tag{4.9.11}$$

then,

$$\mu_1(\kappa, H) \leq \kappa^2\inf_{x\in\Omega}\left\{\frac{H}{\kappa}\,B_0(x) - a(x)\right\} + \mathcal{O}(\kappa)\,, \qquad \text{as}\,\kappa \to +\infty\,. \tag{4.9.12}$$

Proposition 4.9.5 (Upper bound near the boundary)**.** *Suppose that $\Omega \subset \mathbb{R}^2$ is an open bounded set with a smooth boundary, $\lambda_{\max} > 0$ and $\Gamma = \varnothing$. For any $x_0 \in \partial\Omega$ and for any $\sigma \in (0, \lambda_{\max}]$,* *we have,*

$$\mu_{\mathcal{B},\sigma} \leq \frac{\mathcal{B}}{\sigma}\big(\sigma\,\Theta_0\,B_0(x_0) - a(x_0)\big) + \mathcal{O}(\mathcal{B}^{\frac{1}{2}})\,, \qquad \text{as } \mathcal{B}\to+\infty\,. \tag{4.9.13}$$

Here, Θ_0 *is introduced in* (4.8.10).

Proof. We recall the definition of $\mu_{\mathcal{B},\sigma}$ as follows :

$$\mu_{\mathcal{B},\sigma} = \inf_{u\in H^1(\Omega)\setminus\{0\}}\left(\frac{\int_\Omega |(\nabla - i\mathcal{B}\mathbf{F})u(x)|^2\,dx}{\int_\Omega |u(x)|^2\,dx} - \frac{\mathcal{B}}{\sigma}\,\frac{\int_\Omega a(x)|u(x)|^2\,dx}{\int_\Omega |u(x)|^2\,dx}\right)\,.$$

The first term in the right hand side is studied by Helffer-Morame (see [29, Theorem 9.1] with $h=\mathcal{B}^{-1}$ and $\mu_{\mathcal{B},\sigma}=\frac{\mu^{(1)}(h)}{h^2}$) or Fournais-Helffer (see [14, Section 9.2.1]). They proved for any $x_0\in\partial\Omega$ the existence of $\mathcal{B}_0$ such that for $\mathcal{B}\geq\mathcal{B}_0$ one can construct a trial function $\widehat{u}$ such that,

$$\frac{\int_\Omega |(\nabla - i\mathcal{B}\mathbf{F})\widehat{u}(x)|^2\,dx}{\int_\Omega |\widehat{u}(x)|^2\,dx} \leq \mathcal{B}\,\Theta_0\,B_0(x_0) + \mathcal{O}(\mathcal{B}^{\frac{1}{2}})\,, \qquad \text{as } \mathcal{B}\to+\infty\,.$$

The estimates of the second term in the right hand side are just as in (4.9.10) and this achieves the proof of the proposition. □

Remark 4.9.6. $\partial\Omega$ being compact, we can choose $x_0\in\partial\Omega$, such that,

$$\sigma\,\Theta_0\,B_0(x_0) - a(x_0) = \inf_{x\in\partial\Omega}\left\{\sigma\,\Theta_0\,B_0(x) - a(x)\right\}\,,$$

and we apply Proposition 4.9.5 with

$$\mathcal{B}=\kappa H \quad \text{and} \quad \sigma=\frac{H}{\kappa}\,,$$

which implies under Assumption 4.9.11 that,

$$\mu_1(\kappa,H) \leq \kappa^2 \inf_{x\in\partial\Omega}\left\{\frac{H}{\kappa}\,\Theta_0\,B_0(x) - a(x)\right\} + \mathcal{O}(\kappa)\,, \qquad \text{as}\,\kappa\to+\infty\,. \tag{4.9.14}$$

Remarks 4.9.4 and 4.9.6 lead to the conclusion in :

Theorem 4.9.7. *Let* $\Omega\subset\mathbb{R}^2$ *is an open bounded set with a smooth boundary and* $\Gamma=\varnothing$. *Suppose that* (4.9.11) *hold and* $a\in C^1(\overline{\Omega})$, *we have*

$$\mu_1(\kappa,H) \leq \kappa^2\,\Lambda_1\left(B_0, a, \frac{H}{\kappa}\right) + \mathcal{O}(\kappa)\,, \qquad \text{as}\,\kappa\to+\infty\,.$$

Here, Λ_1 *is introduced in* (4.9.1).

Notice that the conclusion in Theorem 4.9.7 is valid under the assumption $\kappa H\geq\mathcal{B}_0$ with $\mathcal{B}_0>0$ sufficiently large. Lemma 4.9.8 below takes care of the regime where $\kappa H=\mathcal{O}(1)$.

Lemma 4.9.8. *Let* $C_{\max}>0$. *Suppose that* $\{a>0\}\neq\emptyset$. *There exists a constant* $\kappa_0>0$ *such that, if*

$$\kappa\geq\kappa_0 \quad \text{and} \quad 0\leq H\leq C_{\max}\kappa^{-1}\,,$$

then

$$\mu_1(\kappa, H) < 0\,.$$

Remark 4.9.9. The conclusion in Lemma 4.9.8 is valid in both cases where $\Gamma = \emptyset$ and $\Gamma \neq \emptyset$.

Proof of Lemma 4.9.8.
Let $\ell > 0$. Choose $x_0 \in \Omega$ such that $a(x_0) > 0$. We introduce a cut-off function $\chi_\ell \in C_c^\infty(\mathbb{R}^2)$ satisfying :

$$0 \leq \chi_\ell \leq 1 \text{ in } \mathbb{R}^2\,, \quad \mathrm{supp}\chi_\ell \subset B(x_0, \ell)\,, \quad \chi_\ell = 1 \text{ in } B\left(x_0, \ell/2\right) \quad \text{and} \quad |\nabla \chi_\ell| \leq \frac{C}{\ell}\,. \tag{4.9.15}$$

The min-max principle yields,

$$\mu^{(1)}(\kappa, H)\|\chi_\ell\|^2_{L^2(\Omega)} \leq \int_\Omega |(\nabla - i\kappa H\mathbf{F})\chi_\ell|^2\,dx - \kappa^2 \int_\Omega a(x)|\chi_\ell(x)|^2\,dx\,.$$

Using the assumptions on χ_ℓ and the fact that $\mathbf{F} \in C^\infty(\overline{\Omega})$, a trivial estimate is,

$$\begin{aligned}\int_\Omega |(\nabla - i\kappa H\mathbf{F})\chi_\ell|^2\,dx &= \int_{B(x_0,\ell)} |\nabla\chi_\ell(x)|^2\,dx + \kappa^2 H^2 \int_{B(x_0,\ell)} |\mathbf{F}\,\chi_\ell(x)|^2\,dx \\ &\leq C\left(1 + (\kappa\, H\, \ell)^2\right).\end{aligned} \tag{4.9.16}$$

We write by Taylor's formula applied to the function a near x_0,

$$-\kappa^2 \int_\Omega a(x)|\chi_\ell(x)|^2\,dx \leq -a(x_0)\,\kappa^2\,\ell^2 + C\,\kappa^2\,\ell^3\,. \tag{4.9.17}$$

Collecting (4.9.16) and (4.9.17), we obtain,

$$\mu^{(1)}(\kappa, H)\|\chi_\ell\|^2_{L^2(\Omega)} \leq -a(x_0)\,\kappa^2\,\ell^2 + C(\kappa^2\,\ell^3 + 1 + (\kappa\, H\,\ell)^2)\,.$$

We select $\ell = \kappa^{-\frac{1}{2}}$ and note that $\kappa H < C_{\max}$. We find that,

$$\mu^{(1)}(\kappa, H)\|\chi_\ell\|^2_{L^2(\Omega)} \leq -a(x_0)\,\kappa + C\Big(\kappa^{\frac{1}{2}} + 1 + C_{\max}^2\kappa^{-1}\Big)\,.$$

Since $\chi_\ell \neq 0$ and $a(x_0) > 0$, we deduce that, for κ sufficiently large,

$$\mu^{(1)}(\kappa, H) < 0\,.$$

□

4.10 Proof of Theorem 4.1.6

4.10.1 Analysis of $H_{C_3}^{loc}$.

In this subsection we give a lower bound of the critical field $\underline{H}_{C_3}^{loc}$ (see (4.1.29)) and we give an upper bound of the critical field $\overline{H}_{C_3}^{loc}$ in the case when the magnetic field B_0 is constant with a pining term.

Proposition 4.10.1. *Suppose that $\{a > 0\} \neq \varnothing$ and $\Gamma = \varnothing$. There exist constants $C > 0$ and $\kappa_0 \geq 0$ such that if*

$$\kappa \geq \kappa_0\,, \qquad H \leq \kappa \max\left(\sup_{x\in\Omega} \frac{a(x)}{B_0(x)}, \sup_{x\in\partial\Omega} \frac{a(x)}{\Theta_0\, B_0(x)}\right) - C\,\kappa^{\frac{1}{2}}\,, \tag{4.10.1}$$

then,

$$\mu_1(\kappa, H) < 0\,.$$

Moreover,

$$\kappa \max\left(\sup_{x\in\Omega} \frac{a(x)}{B_0(x)}, \sup_{x\in\partial\Omega} \frac{a(x)}{\Theta_0\, B_0(x)}\right) - C\,\kappa^{\frac{1}{2}} \leq \underline{H}^{loc}_{C_3}\,.$$

Proof. To apply the previous results, we take

$$\lambda_{max} = \max\left(\sup_{x\in\Omega} \frac{a(x)}{B_0(x)}, \sup_{x\in\partial\Omega} \frac{a(x)}{\Theta_0\, B_0(x)}\right) + 1\,.$$

We have two cases :

Case 1. If

$$\sup_{x\in\Omega} \frac{a(x)}{B_0(x)} > \sup_{x\in\partial\Omega} \frac{a(x)}{\Theta_0\, B_0(x)}\,.$$

then, there exists $x_0 \in \Omega$ (the supremum of $\frac{a(x)}{B_0(x)}$ can not be attained on the boundary, since $\frac{a(x)}{\Theta_0\, B_0(x)} > \frac{a(x)}{B_0(x)}$), such that,

$$\sup_{x\in\Omega} \frac{a(x)}{B_0(x)} = \frac{a(x_0)}{B_0(x_0)}\,.$$

If (4.10.1) is satisfied for some $C > 0$, then,

$$\frac{H}{\kappa} \leq \frac{a(x_0)}{B_0(x_0)} - C\,\kappa^{-\frac{1}{2}}\,,$$

that we can write in the form,

$$\kappa^2\left(\frac{H}{\kappa} B_0(x_0) - a(x_0)\right) \leq -C\,M\,\kappa^{\frac{3}{2}}\,,$$

where $M > 0$ is a constant independent of C.

Suppose that $\kappa H \geq \mathcal{B}_0$ where $\mathcal{B}_0$ is selected sufficiently large such that we can apply Remark 4.9.4. (Thanks to Lemma 4.9.8, $\mu_1(\kappa, H) < 0$ when $\kappa H < \mathcal{B}_0$).

Remark 4.9.4 tells us that there exist positive constants C_1 and κ_0 such that, for $\kappa \geq \kappa_0$,

$$\begin{aligned}
\mu_1(\kappa, H) &\leq \kappa^2 \inf_{x\in\Omega}\left(\frac{H}{\kappa} B_0(x) - a(x)\right) + C_1\,\kappa \\
&\leq \kappa^2\left(\frac{H}{\kappa} B_0(x_0) - a(x_0)\right) + C_1\,\kappa^{\frac{3}{2}} \qquad (4.10.2) \\
&\leq (C_1 - C\,M)\,\kappa^{\frac{3}{2}}\,. \qquad (4.10.3)
\end{aligned}$$

By choosing C such that $C\,M > C_1$, we get,

$$\mu_1(\kappa, H) < 0\,.$$

Case 2. Here, we suppose that

$$\sup_{x\in\partial\Omega}\frac{a(x)}{\Theta_0\,B_0(x)} \geq \sup_{x\in\Omega}\frac{a(x)}{B_0(x)}\,.$$

By compactness, there exists $x_0'\in\partial\Omega$, such that,

$$\sup_{x\in\partial\Omega}\frac{a(x)}{\Theta_0\,B_0(x)} = \frac{a(x_0')}{\Theta_0\,B_0(x_0')}$$

If (4.10.1) is satisfied for some $C>0$, then,

$$\kappa^2\left(\frac{H}{\kappa}\Theta_0\,B_0(x_0') - a(x_0')\right) \leq -C\,M'\,\kappa^{\frac{3}{2}}\,.$$

Thanks to Remark 4.9.6, we get the existence of positive constants C_2 and κ_0 such that, for $\kappa\geq\kappa_0$,

$$\begin{aligned}
\mu_1(\kappa,H) &\leq \kappa^2\inf_{x\in\partial\Omega}\left(\frac{H}{\kappa}\,\Theta_0\,B_0(x) - a(x)\right) + C_2\,\kappa \\
&\leq \kappa^2\left(\frac{H}{\kappa}\,\Theta_0\,B_0(x_0') - a(x_0')\right) + C_2\,\kappa^{\frac{3}{2}} &\quad (4.10.4)\\
&\leq (C_2 - C\,M')\,\kappa^{\frac{3}{2}}\,. &\quad (4.10.5)
\end{aligned}$$

By choosing C such that $C\,M' > C_2$, we get,

$$\mu_1(\kappa, H) < 0\,.$$

This finishes the proof of the proposition. ☐

Proposition 4.10.2. *Suppose that $\{a>0\}\neq\varnothing$, $\lambda_{\max}>0$ and $\Gamma=\varnothing$. There exist constants $C>0$ and $\kappa_0>0$ such that if*

$$\kappa\geq\kappa_0\,,\qquad \lambda_{\max}\,\kappa\geq H > \kappa\,\max\left(\sup_{x\in\Omega}\frac{a(x)}{B_0(x)},\,\sup_{x\in\partial\Omega}\frac{a(x)}{\Theta_0\,B_0(x)}\right) + C\,\kappa^{\frac{1}{2}}\,,\qquad (4.10.6)$$

then,

$$\mu_1(\kappa, H) > 0\,.$$

Moreover,

$$\overline{H}^{loc}_{C_3} \leq \kappa\,\max\left(\sup_{x\in\Omega}\frac{a(x)}{B_0(x)},\,\sup_{x\in\partial\Omega}\frac{a(x)}{\Theta_0\,B_0(x)}\right) + C\,\kappa^{\frac{1}{2}}\,.$$

Proof. To apply the previous results, we take

$$\lambda_{min} = \frac{1}{2} \max \left(\sup_{x\in\Omega} \frac{a(x)}{B_0(x)}, \sup_{x\in\partial\Omega} \frac{a(x)}{\Theta_0\, B_0(x)} \right).$$

If (4.10.6) holds for some $C > 0$, then, for any $x \in \Omega$, we have,

$$\frac{H}{\kappa} B_0(x) - a(x) \geq C\,\kappa^{-\frac{1}{2}}\,, \tag{4.10.7}$$

and, for any $x' \in \partial\Omega$, we have,

$$\frac{H}{\kappa} \Theta_0 B_0(x') - a(x') \geq C\,\kappa^{-\frac{1}{2}}\,. \tag{4.10.8}$$

Having in mind the definition of Λ_1 in (4.9.1), the estimates in (4.10.7) and in (4.10.8) give us that for κ large enough,

$$\Lambda_1 \left(B_0, a, \frac{H}{\kappa} \right) \geq C\,\kappa^{-\frac{1}{2}}\,.$$

Thanks to Theorem 4.9.2, we get the existence of positive constants C' and κ_0 such that, for $\kappa \geq \kappa_0$,

$$\begin{aligned} \mu_1(\kappa, H) &\geq \kappa^2\,\Lambda_1 \left(B_0, a, \frac{H}{\kappa} \right) - C'\,\kappa^{\frac{3}{2}} \\ &\geq (C - C')\,\kappa^{\frac{3}{2}}\,. \end{aligned} \tag{4.10.9}$$

To finish this proof, we choose $C > C'$. □

As a consequence, we have proved Theorem 4.1.6 for $\underline{H}^{loc}_{C_3}$ and $\overline{H}^{loc}_{C_3}$

4.10.2 Analysis of $H^{cp}_{C_3}$

In this subsection we give a lower bound of the critical field $\underline{H}^{cp}_{C_3}$ (see (4.1.27)) and we give an upper bound of the critical field $\overline{H}^{cp}_{C_3}$ in the case when the magnetic field B_0 is constant with a pining term. We start with a proposition which measures the effect of the localization at the boundary when H is sufficiently large.

Proposition 4.10.3. *Suppose that $\Gamma = \varnothing$ and* (4.10.6) *holds. There exists a positive constant C, such that if $(\psi, \mathbf{A})$ is a solution of* (4.1.12)*, then the following estimate holds :*

$$\|\psi\|^2_{L^2(\Omega)} \leq C\,\kappa^{-\frac{3}{8}} \|\psi\|^2_{L^4(\Omega)}\,. \tag{4.10.10}$$

Proof.
The techniques that will be used in this proof are similar with the ones in [18, Lemma 2.6]. If H satisfies (4.10.6) for some $C > 0$, then, for any $x \in \Omega$, we have.

$$\kappa\,H\,B_0(x) - \kappa^2\,a(x) \geq C\,\kappa^{\frac{3}{2}}\,. \tag{4.10.11}$$

First, we let $\chi \in C^\infty(\mathbb{R})$ be a standard cut-off function such that

$$\chi = 1 \quad \text{in } [1,\infty] \qquad \text{and} \qquad \chi = 0 \quad \text{in }]-\infty, 1/2]\,. \tag{4.10.12}$$

Next, we define $\lambda = \kappa^{-\frac{3}{4}}$, and χ_κ as follows :

$$\chi_\kappa(x) = \chi\left(\frac{\operatorname{dist}(x,\partial\Omega)}{\lambda}\right)\,, \qquad \forall x \in \Omega\,. \tag{4.10.13}$$

Referring to (4.7.6), we have

$$\int_\Omega \left(|(\nabla - i\kappa H\mathbf{A})\chi_\kappa\psi|^2 - |\nabla\chi_\kappa|^2|\psi|^2\right)\,dx = \kappa^2\int_\Omega |\chi_\kappa|^2(a(x)-|\psi|^2)|\psi|^2\,dx\,. \tag{4.10.14}$$

We estimate $\int_\Omega |(\nabla - i\kappa H\mathbf{A})\chi_\kappa\psi|^2\,dx$ from below. As in [27, Proposition 6.2], we can prove that,

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A})\chi_\kappa\psi|^2\,dx \ge \kappa\,H\int_\Omega \operatorname{curl}\mathbf{F}\,|\chi_\kappa\psi|^2\,dx - \kappa\,H\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)}\|\chi_\kappa\psi\|^2_{L^4(\Omega)}\,.$$

Noticing that $\operatorname{curl}\mathbf{F} = B_0(x)$ and $\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)} \le \dfrac{c}{H}\|\psi\|_{L^2(\Omega)}$, we have,

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A})\chi_\kappa\psi|^2\,dx \ge \kappa\,H\int_\Omega B_0(x)\,|\chi_\kappa\psi|^2\,dx - c\,\kappa\,\|\psi\|_{L^2(\Omega)}\|\chi_\kappa\psi\|^2_{L^4(\Omega)}\,.$$

Implementing a Cauchy-Schwarz inequality, we get

$$\int_\Omega |(\nabla - i\kappa H\mathbf{A})\chi_\kappa\psi|^2\,dx \ge \kappa\,H\int_\Omega B_0(x)\,|\chi_\kappa\psi|^2\,dx - c^2\,\|\psi\|^2_{L^2(\Omega)} - \kappa^2\|\chi_\kappa\psi\|^4_{L^4(\Omega)}\,. \tag{4.10.15}$$

Inserting (4.10.15) into (4.10.14), we obtain,

$$\int_\Omega \left(\kappa\,H\,B_0(x) - \kappa^2\,a(x)\right)\,|\chi_\kappa\psi|^2\,dx \le c^2\int_\Omega |\psi|^2\,dx + \int_\Omega |\nabla\chi_\kappa|^2|\psi|^2\,dx - \kappa^2\int_\Omega \left(\chi_\kappa^2 - \chi_\kappa^4\right)|\psi|^4\,dx\,.$$

As a consequence of (4.10.11), the inequality above becomes,

$$C\,\kappa^{\frac{3}{2}}\int |\chi_\kappa\psi(x)|^2\,dx \le c^2\int_\Omega |\psi|^2\,dx + \int_\Omega |\nabla\chi_\kappa|^2|\psi|^2\,dx - \kappa^2\int_\Omega \left(\chi_\kappa^2-\chi_\kappa^4\right)|\psi|^4\,dx\,.$$

Notice that $-\kappa^2\int_\Omega \left(\chi_\kappa^2-\chi_\kappa^4\right)|\psi|^4\,dx \le 0\,.$
Decomposing the integral $\displaystyle\int_\Omega |\psi|^2\,dx = \int_\Omega |\chi_\kappa\psi|^2\,dx + \int_\Omega (1-\chi_\kappa^2)|\psi|^2\,dx$, using (4.10.11) and choosing C such that $C \ge 2c^2$, we get,

$$\frac{1}{2}C\,\kappa^{\frac{3}{2}}\int |\chi_\kappa\psi(x)|^2\,dx \le \left(c^2 + \|\chi'\|^2_{L^\infty(\mathbb{R})}\,\lambda^{-2}\right)\int_{\{x\in\Omega:\,\operatorname{dist}(x,\Gamma)\le\lambda\}} |\psi|^2\,dx\,.$$

Recall that $\lambda = \kappa^{-\frac{3}{4}}$, we observe that,

$$\int |\chi_\kappa \psi(x)|^2 \, dx \le 4\|\chi'\|^2_{L^\infty(\mathbb{R})} \int_{\{x\in\Omega:\, \mathrm{dist}(x,\Gamma)\le\lambda\}} |\psi|^2 \, dx \,,$$

and consequently, we get,

$$\int |\psi(x)|^2 \, dx \le \left(4\|\chi'\|^2_{L^\infty(\mathbb{R})} + 1\right) \int_{\{x\in\Omega:\, \mathrm{dist}(x,\Gamma)\le\lambda\}} |\psi|^2 \, dx \,.$$

By choosing $C = \max\left(2\,c^2, 4\|\chi'\|^2_{L^\infty(\mathbb{R})} + 1\right)$, we obtain,

$$\|\psi\|^2_{L^2(\Omega)} \le C\,\kappa^{-\frac{3}{8}}\|\psi\|^2_{L^4(\Omega)} \,.$$

□

Theorem 4.10.4. *Supose that $\Gamma = \varnothing$ and $\{a > 0\} \neq \varnothing$. There exists $C > 0$ and κ_0 such that, if H satisfies*

$$H > \kappa \max\left(\sup_{x\in\Omega} \frac{a(x)}{B_0(x)}, \sup_{x\in\partial\Omega} \frac{a(x)}{\Theta_0\, B_0(x)}\right) + C\,\kappa^{\frac{1}{2}} \,, \tag{4.10.16}$$

then $(0, \mathbf{F})$ is the unique solution to (4.1.12).
Moreover,

$$\overline{H}^{cp}_{C_3} \le \kappa \max\left(\sup_{x\in\Omega} \frac{a(x)}{B_0(x)}, \sup_{x\in\partial\Omega} \frac{a(x)}{\Theta_0\, B_0(x)}\right) + C\,\kappa^{\frac{1}{2}} \,.$$

Proof. We first observe that it results from Giorgi-Phillips like Theorem 4.8.3 that it remains only to prove the theorem under the stronger Assumption (4.10.6). Suppose now that $(\psi, \mathbf{A})$ is a solution of (4.1.12) with $\psi \neq 0$, we observe that,

$$0 < \kappa^2\|\psi\|^4_{L^4(\Omega)} = -\int_\Omega \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2 a(x)|\psi|^2\right) \, dx := \top \,. \tag{4.10.17}$$

We can write,

$$\begin{aligned} -\top &\ge (1 - \sqrt{\top}\,\kappa^{-1}) \int_\Omega |(\nabla - i\kappa H\mathbf{F})\psi|^2 \, dx - \kappa^2 \int_\Omega a(x)|\psi|^2 \, dx - \frac{(\kappa H)^2}{\sqrt{\top}\kappa^{-1}} \int_\Omega |(\mathbf{A} - \mathbf{F})\psi|^2 \, dx \\ &\ge \mu_1(\kappa, H)\,\|\psi\|^2_{L^2(\Omega)} - \sqrt{\top}\,\kappa^{-1}\|(\nabla - i\kappa H\mathbf{F})\psi\|^2_{L^2(\Omega)} - \frac{(\kappa H)^2}{\sqrt{\top}\kappa^{-1}}\|(\mathbf{A} - \mathbf{F})\psi\|^2_{L^2(\Omega)} \,. \end{aligned} \tag{4.10.18}$$

We reffer to (4.8.3) and (4.8.7), we have,

$$-\top \ge \mu_1(\kappa, H)\,\|\psi\|^2_{L^2(\Omega)} - C\,\sqrt{\top}\,\kappa\,\|\psi\|^2_{L^2(\Omega)} \,. \tag{4.10.19}$$

Thanks to Proposition 4.10.3, using (4.10.17), we get,

$$\|\psi\|^2_{L^2(\Omega)} \le C\,\kappa^{-\frac{11}{8}}\,\sqrt{\top} \,. \tag{4.10.20}$$

As a consequence of (4.10.20), (4.10.19) becomes,

$$-\top \geq \mu_1(\kappa, H)\,\|\psi\|^2_{L^2(\Omega)} - C'\,\kappa^{-\frac{3}{8}}\,\top\,. \tag{4.10.21}$$

Having in mind that $\psi \neq 0$ and $\top > 0$ (see (4.10.17)), we deduce for κ sufficiently large $\mu_1(\kappa, H) < 0$, which is in contradiction with Proposition 4.10.2. Therefore, we conclude that $\psi = 0$, which is what we needed to prove. □

Proposition 4.10.5. *Supose that* $\Gamma = \varnothing$ *and* $\{a > 0\} \neq \varnothing$. *There exists* $C > 0$ *and* κ_0 *such that, if* H *satisfies*

$$H \leq \kappa \max\left(\sup_{x\in\Omega}\frac{a(x)}{B_0(x)}, \sup_{x\in\partial\Omega}\frac{a(x)}{\Theta_0\,B_0(x)}\right) - C\,\kappa^{\frac{1}{2}}\,, \tag{4.10.22}$$

then there exists a solution $(\psi, \mathbf{A})$ *of* (4.1.12) *with* $\|\psi\|_{L^2(\Omega)} \neq 0$.
Moreover,

$$\kappa \max\left(\sup_{x\in\Omega}\frac{a(x)}{B_0(x)}, \sup_{x\in\partial\Omega}\frac{a(x)}{\Theta_0\,B_0(x)}\right) - C\,\kappa^{\frac{1}{2}} \leq \underline{H}^{cp}_{C_3}\,.$$

Proof. We use $(t\psi_*, \mathbf{F})$, with t sufficiently small and ψ_* an eigenfunction associated with $\mu_1(\kappa, H)$, as a test configuration for the functional (4.1.1), i.e.

$$\int_\Omega \left(|(\nabla - i\kappa H\mathbf{F})\psi_*|^2 - \kappa^2\,a(x)|\psi_*|^2\right)\,dx = \mu_1(\kappa, H)\|\psi_*\|^2_{L^2(\Omega)}\,.$$

Proposition 4.10.1 tells us that there exists a constant C such that, under Assumption (4.10.22), $\mu_1(\kappa, H) < 0\,.$
Therefore,

$$C_1(\kappa, H) := \int_\Omega \left(|(\nabla - i\kappa H\mathbf{F})\psi_*|^2 - \kappa^2\,a(x)|\psi_*|^2\right)\,dx < 0\,.$$

We can write,

$$\begin{aligned}\mathcal{E}_{\kappa,H,a,B_0}(t\psi_*, \mathbf{F}) &= t^2 \int_\Omega \left(|(\nabla - i\kappa H\mathbf{F})\psi_*|^2 - \kappa^2\,a(x)|\psi_*|^2\right)\,dx + t^4\,\frac{\kappa^2}{2}\int_\Omega |\psi_*|^4\,dx + \frac{\kappa^2}{2}\int_\Omega a(x)\,dx\\ &= t^2\left(C_1(\kappa, H) + t^2\,\frac{\kappa^2}{2}\int_\Omega |\psi_*|^4\,dx\right) + \mathcal{E}_{\kappa,H,a,B_0}(0, \mathbf{F})\,.\end{aligned}$$

We choose t such that,

$$C_1(\kappa, H) + t^2\,\frac{\kappa^2}{2}\int_\Omega |\psi_*|^4\,dx < 0\,.$$

Thus, we get

$$\mathcal{E}_{\kappa,H,a,B_0}(t\psi_*, \mathbf{F}) < \mathcal{E}_{\kappa,H,a,B_0}(0, \mathbf{F})\,.$$

Hence a minimizer, which is a solution of (4.1.12), will be non-trivial. □

4.10.3 End of the proof of Theorem 4.1.6

First, we will prove the following inclusion,

$$\mathcal{N}^{\mathrm{loc}}(\kappa) \subset \mathcal{N}(\kappa)\,.$$

We see that if $H \notin \mathcal{N}(\kappa)$, then $(0, \mathbf{F})$ is a local minimizer of $\mathcal{E}_{\kappa,H,a,B_0}$. Thus, the Hessian of the functional $\mathcal{E}_{\kappa,H,a,B_0}$ at the normal state $(0, \mathbf{F})$ should be positive.
For every $(\widetilde{\phi}, \widetilde{\mathbf{A}}) \in H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)$ we have,

$$\mathcal{E}_{\kappa,H,a,B_0}(t\widetilde{\phi}, \mathbf{F} + t\widetilde{\mathbf{A}}) = t^2 \left[\mathcal{Q}^{\Omega}_{\kappa H\mathbf{F},-\kappa^2 a}(\widetilde{\phi}) + (\kappa H)^2 \int_{\Omega} |\operatorname{curl} \widetilde{\mathbf{A}}|^2\, dx \right] + \mathcal{O}(t^3)\,.$$

This implies that the Hessian of the functional $\mathcal{E}_{\kappa,H,a,B_0}$ at the normal state $(0, \mathbf{F})$ can be written as follows :

$$Hess_{(0,\mathbf{F})}[\widetilde{\phi}, \widetilde{\mathbf{A}}] = \mathcal{Q}^{\Omega}_{\kappa H\mathbf{F},-\kappa^2 a}(\widetilde{\phi}) + (\kappa H)^2 \int_{\Omega} |\operatorname{curl} \widetilde{\mathbf{A}}|^2\, dx\,.$$

Since $Hess_{(0,\mathbf{F})}[\widetilde{\phi}, \widetilde{\mathbf{A}}] \geq 0$, we get that $\mu_1(\kappa H) \geq 0$, and consequently $H \notin \mathcal{N}^{\mathrm{loc}}(\kappa)$. Hence we obtain the above inclusion.
On the other hand, if $(\psi, \mathbf{A})$ is a minimizer of the functional in (4.1.1) with $\psi \neq 0$, then $(\psi, \mathbf{A})$ is a solution of (4.1.12), and we have the following inclusion,

$$\mathcal{N}(\kappa) \subset \mathcal{N}^{\mathrm{cp}}(\kappa)\,,$$

and consequently,

$$\mathcal{N}^{\mathrm{loc}}(\kappa) \subset \mathcal{N}(\kappa) \subset \mathcal{N}^{\mathrm{cp}}(\kappa)\,. \tag{4.10.23}$$

Having in mind the definition of all the critical fields in (4.1.27), (4.1.28) and (4.1.29), we deduce that,

$$\overline{H}^{loc}_{C_3}(\kappa) \leq \overline{H}_{C_3}(\kappa) \leq \overline{H}^{cp}_{C_3}(\kappa)\,, \tag{4.10.24}$$

Using (4.10.23), we observe that,

$$\mathbb{R}^+ \setminus \mathcal{N}^{\mathrm{cp}}(\kappa) \subset \mathbb{R}^+ \setminus \mathcal{N}(\kappa) \subset \mathbb{R}^+ \setminus \mathcal{N}^{\mathrm{loc}}(\kappa)\,.$$

From the definition of all the critical fields, we conclude that,

$$\underline{H}^{loc}_{C_3}(\kappa) \leq \underline{H}_{C_3}(\kappa) \leq \underline{H}^{cp}_{C_3}(\kappa)\,. \tag{4.10.25}$$

We note that $\underline{H}^{loc}_{C_3} \leq \overline{H}^{loc}_{C_3}$ and $\underline{H}^{cp}_{C_3} \leq \overline{H}^{cp}_{C_3}$. Therefore, all the critical fields are contained in the interval $[\underline{H}^{loc}_{C_3}, \overline{H}^{cp}_{C_3}]$.
By Proposition 4.10.1 and Theorem 4.10.4, we get the existence of positive constants C and κ_0,

such that for $\kappa \geq \kappa_0$,

$$\kappa \max\left(\sup_{x\in\Omega}\frac{a(x)}{B_0(x)}, \sup_{x\in\partial\Omega}\frac{a(x)}{\Theta_0\, B_0(x)}\right) - C\,\kappa^{\frac{1}{2}} \leq \underline{H}_{C_3}^{loc} \leq \overline{H}_{C_3}^{cp}$$
$$\leq \kappa \max\left(\sup_{x\in\Omega}\frac{a(x)}{B_0(x)}, \sup_{x\in\partial\Omega}\frac{a(x)}{\Theta_0\, B_0(x)}\right) + C\,\kappa^{\frac{1}{2}}\,. \quad (4.10.26)$$

As a consequence, we have proved Theorem 4.1.6 for the six critical fields.

Remark 4.10.6. As in [14], it would be interesting to show that all the critical fields coincide when κ is large enough.

4.11 Asymptotics of $\mu_1(\kappa, H)$: the case with vanishing magnetic field

In this section we give an estimate for the lowest eigenvalue $\mu_1(\kappa, H)$ of the operator $P^{\Omega}_{\kappa H\mathbf{F},-\kappa^2 a}$ (see (4.1.26)) in the case when $\Gamma = \varnothing$ with a κ-independent pinning, i.e. $a(\kappa, x) = a(x)$. The results in this section are valid under the assumption $\Gamma \neq \emptyset$, where the set Γ is introduced in (4.1.3). Let

$$\mathcal{B} = \kappa H \qquad \text{and} \qquad \widehat{\sigma} = \frac{H}{\kappa^2}\,. \quad (4.11.1)$$

We observe that,

$$P^{\Omega}_{\kappa H\mathbf{F},-\kappa^2 a} = P^{\Omega}_{\mathcal{B}\mathbf{F},-\left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}} a}\,.$$

We will give an estimate for the lowest eigenvalue $\mu_{\mathcal{B},\widehat{\sigma}}$ of $P^{\Omega}_{\mathcal{B}\mathbf{F},-\left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}} a}$. After a change of notation, we deduce an estimate for $\mu_1(\kappa, H)$.

4.11.1 Lower bound

In the absence of a pinning term, that is when $a = 1$, Pan and Kwek [42] gave the lower bound for the lowest eigenvalue $\mu(\mathcal{B}\mathbf{F})$ of $P^{\Omega}_{\mathcal{B}\mathbf{F},0}$ when $\mathcal{B} \to +\infty$. In this subsection, we determine a lower bound for μ_1 when $\kappa \to +\infty$ and the pinning term is present.
We first recall the definition of λ_0 in (4.1.31), the definition of Γ in (4.1.3) and for any $\theta \in (0, \pi)$ we recall that $\lambda(\mathbb{R}^2_+, \theta)$ is the bottom of the spectrum of the operator $P^{\mathbb{R}^2_+}_{\mathbf{A}_{\mathrm{app},\theta},0}$, with

$$\mathbf{A}_{\mathrm{app},\theta} = -\left(\frac{x_2^2}{2}\cos\,\theta, \frac{x_1^2}{2}\sin\,\theta\right)\,.$$

We then define for any $\widehat{\sigma} > 0$,

$$\widehat{\Lambda}_1(B_0, a, \widehat{\sigma}) = \min\left\{\inf_{x\in\Gamma\cap\Omega}\left\{\lambda_0\left(\widehat{\sigma}\,|\nabla B_0(x)|\right)^{\frac{2}{3}} - a(x)\right\},\right.$$
$$\left.\inf_{x\in\Gamma\cap\partial\Omega}\left\{\lambda(\mathbb{R}^2_+, \theta(x))\left(\widehat{\sigma}\,|\nabla B_0(x)|\right)^{\frac{2}{3}} - a(x)\right\}\right\}\,. \quad (4.11.2)$$

Here, for $x \in \Gamma \cap \partial\Omega$, $\theta(x)$ denotes the angle between $\nabla B_0(x)$ and the inward normal vector $-\nu(x)$.

We start with a proposition that states a lower bound of $\mu_1(\kappa, H)$ in the case when $\Gamma \neq \varnothing$.

Proposition 4.11.1. *Let I be a closed interval in $(0,\infty)$. There exist two positive constants $\mathcal{B}_0 > 0$ and $C > 0$ such that if $\mathcal{B} \geq \mathcal{B}_0$, $\widehat{\sigma} \in I$, $\psi \in H^1(\Omega) \setminus \{0\}$ and $a \in C^1(\overline{\Omega})$, then,*

$$\frac{\mathcal{Q}^{\Omega}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(\psi)}{\|\psi\|^2_{L^2(\Omega)}} \geq \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}} \left(\widehat{\Lambda}_1(B_0, a, \widehat{\sigma}) - C\mathcal{B}^{-\frac{1}{18}}\right). \tag{4.11.3}$$

Proof. Let $\ell = B^{-7/29}$. We define the following sets,

$$D_1 = \{x \in \Omega : \operatorname{dist}(x,\Gamma) < 2\,\ell\}\,, \qquad D_2 = \{x \in \Omega : \operatorname{dist}(x,\Gamma) > \ell\}\,.$$

Let h_j be a partition of unity satisfying

$$\sum_{j=1}^{2} h_j^2 = 1\,, \qquad \sum_{j=1}^{2} |\nabla h_j|^2 \leq C\,\ell^{-2} = C\mathcal{B}^{14/29} \qquad \text{and} \qquad \operatorname{supp} h_j \subset D_j \quad (j \in \{1,2\})\,.$$

There holds the following decomposition,

$$\begin{aligned}
\mathcal{Q}^{\Omega}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(\psi) &= \mathcal{Q}^{D_1}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(h_1\psi) + \mathcal{Q}^{D_2}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(h_2\psi) - \sum_{j=1}^{2}\int_\Omega |\nabla h_j|^2|\psi|^2\,dx\\
&\geq \mathcal{Q}^{D_1}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(h_1\psi) + \mathcal{Q}^{D_2}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(h_2\psi) - C\mathcal{B}^{14/29}\int_\Omega |\psi|^2\,dx\,.
\end{aligned}\tag{4.11.4}$$

We cover the curve Γ by a family of disks

$$D(\omega_j,\ell) \subset \{x \in \mathbb{R}^2 : \operatorname{dist}(x,\Gamma) \leq 2\ell\} \qquad \text{and} \qquad D_1 \subset \bigcup_j D(\omega_j,\ell) \qquad (\omega_j \in \Gamma)\,.$$

Consider a partition of unity satisfying

$$\sum_j \chi_j^2 = 1\,, \qquad \sum_j |\nabla\chi_j|^2 \leq C\,\ell^{-2} \qquad \text{and} \qquad \operatorname{supp}\chi_j \subset D(\omega_j,\ell)\,.$$

Moreover, we can add the property that :

$$\text{either } \operatorname{supp}\chi_j \cap \Gamma \cap \partial\Omega = \varnothing\,, \quad \text{either } \omega_j \in \Gamma \cap \partial\Omega\,.$$

We may write,

$$\mathcal{Q}^{D_1}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(h_1\psi) = \sum_{int} \mathcal{Q}^{D_1}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(\chi_j h_1\psi) + \sum_{bnd} \mathcal{Q}^{D_1}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(\chi_j h_1\psi) - \sum_j \int_{D_1} |\nabla\chi_j|^2 |h_1\psi|^2\,dx\,, \tag{4.11.5}$$

where 'int' is in reference to the j's such that $\omega_j \in \Gamma \cap \Omega$ and 'bnd' is in reference to the j's such

that $\omega_j \in \Gamma \cap \partial\Omega$.
For the last term on the right side of (4.11.5), we get using the assumption on χ_j :

$$\int_{D_1} |\nabla \chi_j|^2 |h_1\psi|^2\, dx \leq C\,\ell^{-2} \int_{D_1} |h_1\psi|^2\, dx = C\,\mathcal{B}^{14/29} \int_{D_1} |h_1\psi|^2\, dx\,. \tag{4.11.6}$$

We have to find a lower bound for $\mathcal{Q}^{D_1}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(h_1\psi)$ for each j such that $\omega_j \in \Gamma \cap \Omega$ and for each j such that $\omega_j \in \Gamma \cap \partial\Omega$. Thanks to [38], we have,

$$\int_\Omega |(\nabla - i\mathcal{B}\mathbf{F})\chi_j h_1\psi|^2\, dx \geq \mathcal{B}^{\frac{2}{3}} \int_\Omega \Big((\lambda_0\, |\nabla B_0(\omega_j)|\Big)^{\frac{2}{3}} - C B^{-1/18} \Big) |\chi_j h_1 \psi|^2\, dx\,.$$

Using Taylor's formula, we can write in every disk $D(w_j,\ell)$,

$$|a(x) - a(w_j)| \leq C\ell = C\mathcal{B}^{-7/29} \leq C\mathcal{B}^{-1/18}\,. \tag{4.11.7}$$

In that way, we get,

$$\begin{aligned}
&\sum_{int} \mathcal{Q}^{D_1}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(\chi_j h_1\psi) \\
&\quad\geq \sum_{int} \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}} \left(\lambda_0 \left(\widehat{\sigma}\,|\nabla B_0(\omega_j)|\right)^{\frac{2}{3}} - a(\omega_j) - C\mathcal{B}^{-1/18} \right) \int |\chi_j h_1\psi|^2\, dx \\
&\quad\geq \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}} \left(\inf_{x\in\Gamma\cap\Omega} \left\{ \lambda_0 \left(\widehat{\sigma}\,|\nabla B_0(x)|\right)^{\frac{2}{3}} - a(x) \right\} - C\mathcal{B}^{-1/18} \right) \sum_{int} \int |\chi_j h_1\psi|^2\, dx\,.
\end{aligned} \tag{4.11.8}$$

In a similar fashion, the analysis in [38] and (4.11.7) yields,

$$\begin{aligned}
&\sum_{bnd} \mathcal{Q}^{D_1}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(\chi_j h_1\psi) \\
&\quad\geq \sum_{bnd} \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}} \left(\lambda(\mathbb{R}^2_+,\theta(\omega_j)) \left(\widehat{\sigma}\,|\nabla B_0(\omega_j)|\right)^{\frac{2}{3}} - a(\omega_j) - C\mathcal{B}^{-1/18} \right) \int |\chi_j h_1\psi|^2\, dx \\
&\quad\geq \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}} \left(\inf_{x\in\Gamma\cap\partial\Omega} \left\{ \lambda(\mathbb{R}^2_+,\theta(x)) \left(\widehat{\sigma}\,|\nabla B_0(x)|\right)^{\frac{2}{3}} - a(x) \right\} - C\mathcal{B}^{-1/18} \right) \sum_{bnd} \int |\chi_j h_1\psi|^2\, dx\,.
\end{aligned} \tag{4.11.9}$$

We insert (4.11.8), (4.11.9) and (4.11.6) into (4.11.5) to obtain,

$$\mathcal{Q}^{D_1}_{\mathcal{B}\mathbf{F},-(\frac{\mathcal{B}}{\widehat{\sigma}})^{\frac{2}{3}}a}(h_1\psi) \geq \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}} \left(\widehat{\Lambda}_1(B_0,a,\widehat{\sigma}) - C\mathcal{B}^{-1/18} \right) \int |h_1\psi|^2\, dx\,. \tag{4.11.10}$$

Now, we will bound $\int_\Omega |(\nabla - i\mathcal{B}\mathbf{F})h_2\psi|^2\, dx$ from below. Let $\ell_1 < \ell$, we cover D_2 by a family of disks

$$D(\omega'_j,\ell_1) \subset \{x\in\mathbb{R}^2 : \mathrm{dist}(x,\Gamma) \geq \ell_1\} \qquad \left(\omega'_j \in \overline{\Omega}\right)\,.$$

Consider a partition of unity satisfying

$$\sum_j \chi_j^2 = 1\,, \qquad \sum_j |\nabla \chi_j|^2 \leq C\,\ell_1^{-2} \qquad \text{and} \qquad \operatorname{supp}\chi_j \subset D(\omega_j', \ell_1)\,.$$

There holds the decomposition formula,

$$\begin{aligned}\int_\Omega |(\nabla - i\mathcal{B}\mathbf{F})h_2\psi|^2\,dx &= \sum_j \int_\Omega |(\nabla - i\mathcal{B}\mathbf{F})\chi_j\, h_2\psi|^2\,dx - \sum_j \int_\Omega |\nabla \chi_j|^2 |h_2\psi|^2\,dx \\ &\geq \sum_j \int_\Omega |(\nabla - i\mathcal{B}\mathbf{F})\chi_j\, h_2\psi|^2\,dx - C\ell_1^{-2}\int_\Omega |h_2\psi|^2\,dx\,, \end{aligned} \tag{4.11.11}$$

We observe that there exists a gauge function φ_j satisfying (see [5, Equation (A.3)]),

$$\big|\mathbf{F}(x) - (B_0(\omega_j')\mathbf{A}_0(x-\omega_j') + \nabla\varphi_j)\big| \leq C\,\ell_1^2 \quad \text{in } D(\omega_j', \ell_1')\,.$$

Using Cauchy-Schwarz inequality, we may write,

$$\begin{aligned}\int_\Omega |(\nabla - i\mathcal{B}\mathbf{F})\chi_j\, h_2\psi|^2\,dx \geq \frac{1}{2}\int_\Omega |(\nabla - i\mathcal{B}\,B_0(\omega_j')\mathbf{A}_0(x-\omega_j'))e^{-i\mathcal{B}\varphi_j}\chi_j\, h_2\psi|^2\,dx \\ - C\,\mathcal{B}^2\,\ell_1^4\int_\Omega |\chi_j\, h_2\psi|^2\,dx\,.\end{aligned}$$

We are reduced to the analysis of the Neumann realization of the Schrödinger operator with a constant magnetic field equal to $\mathcal{B}\,B_0(\omega_j')$ in our case.
Notice that by the assumption on h_2, there exist constants $M > 0$ and $\mathcal{B}_0 > 0$ such that, for all j, $|B_0(\omega_j')| \geq M\,\ell$ in the support of h_2. Thus,

$$\forall j, \quad \mathcal{B}|B_0(\omega_j')| \geq M\,\mathcal{B}\,\ell = M\mathcal{B}^{22/29} \gg 1\,.$$

Moreover, the magnetic potentials $\mathbf{A}_0(x)$ and $\mathbf{A}_0(x-\omega_j')$ are gauge equivalent since

$$\mathbf{A}_0(x-\omega_j') = \mathbf{A}_0(x) - \mathbf{A}_0(\omega_j') = \mathbf{A}_0(x) - \nabla(\mathbf{A}_0(\omega_j')\cdot x)\,.$$

Thanks to Theorem 4.8.2, there exists a constant $\mathcal{B}_0$ such that, for any $\mathcal{B} \geq \mathcal{B}_0$, we write by the min-max principle,

$$\begin{aligned}\sum_j \int_\Omega |(\nabla - i\mathcal{B}\mathbf{F})\chi_j\, h_2\psi|^2\,dx &\geq \frac{\Theta_0\mathcal{B}\,|B_0(\omega_j')|}{2}\sum_{int}\int_\Omega |\chi_j\, h_2\psi|^2\,dx - C\,\mathcal{B}^2\,\ell_1^4 \sum_{int}\int_\Omega |\chi_j\, h_2\psi|^2\,dx \\ &\geq \left(\frac{M\Theta_0}{2}\mathcal{B}\,\ell - C\mathcal{B}^2\,\ell_1^4\right)\sum_j \int_\Omega |\chi_j\, h_2\psi|^2\,dx \\ &= \left(\frac{M\Theta_0}{2}\mathcal{B}\,\ell - C\mathcal{B}^2\,\ell_1^4\right)\int_\Omega |h_2\psi|^2\,dx\,.\end{aligned} \tag{4.11.12}$$

Putting (4.11.12) into (4.11.11), we obtain

$$\begin{aligned}\mathcal{Q}^{D_2}_{\mathcal{B}\mathbf{F},-\left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}}a}(h_2\psi) &= \int_\Omega |(\nabla - i\mathcal{B}\mathbf{F})h_2\psi|^2\,dx - \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{2/3}\int_\Omega a(x)|h_2\psi|^2\,dx\\ &\geq \left(\frac{M\Theta_0}{2}\mathcal{B}\,\ell - C\mathcal{B}^2\,\ell_1^4 - C\ell_1^{-2}\right)\int_\Omega |h_2\psi|^2\,dx - \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{2/3}\int_\Omega a(x)|h_2\psi|^2\,dx\,.\end{aligned} \tag{4.11.13}$$

We choose $\ell_1 = B^{-\rho}$ and $\frac{9}{22} < \rho < \frac{11}{29}$. We observe that,

$$\mathcal{B}^2\,\ell_1^4 = \mathcal{B}^{2-4\rho} \ll \mathcal{B}^{22/29} = \mathcal{B}\,\ell\,,\quad \ell_1^{-2} = B^{2\rho} \ll \mathcal{B}\,\ell\,,\quad \mathcal{B}^{2/3} \ll \mathcal{B}^{22/29} = \mathcal{B}\,\ell\,.$$

In this way, we infer from (4.11.13), that there exists a constant $c > 0$ such that, for $\mathcal{B}$ sufficiently large,

$$\mathcal{Q}^{D_2}_{\mathcal{B}\mathbf{F},-\left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}}a}(h_2\psi) \geq c\mathcal{B}^{22/9}\int_\Omega |h_2\psi|^2\,dx \geq \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}}\widehat{\Lambda}_1(B_0, a, \widehat{\sigma})\int_\Omega |h_2\psi|^2\,dx\,. \tag{4.11.14}$$

Collecting (4.11.4), (4.11.10) and (4.11.14), we finish the proof of Proposition 4.11.1. □

Theorem 4.11.2 is valid under the assumption that,

$$\widehat{\lambda}_{\min} \leq \frac{H}{\kappa^2} \leq \widehat{\lambda}_{\max}\,, \tag{4.11.15}$$

where $0 < \widehat{\lambda}_{\min} < \widehat{\lambda}_{\max} < \infty$ are constants independent of κ and H.

Theorem 4.11.2. *Let $\Omega \subset \mathbb{R}^2$ is an open bounded set with a smooth boundary and $\Gamma \neq \varnothing$. Suppose that (4.11.15) hold and $a \in C^1(\overline{\Omega})$, we have*

$$\mu_1(\kappa, H) \geq \kappa^2\,\widehat{\Lambda}_1\left(B_0, a, \frac{H}{\kappa^2}\right) + \mathcal{O}(\kappa^{\frac{11}{6}})\,,\qquad \text{as}\,\kappa \to +\infty\,.$$

Here, $\widehat{\Lambda}_1$ is introduced in (4.11.2).

Proof. We apply Proposition 4.11.1 with

$$\mathcal{B} = \kappa H\,,\quad \widehat{\sigma} = \frac{H}{\kappa^2}\quad \text{and}\quad I = [\widehat{\lambda}_{\min}, \widehat{\lambda}_{\max}]\,.$$

Let us verify that the conditions of the proposition are satisfied for this choice. Thanks to (4.11.15), $\widehat{\sigma} \in I$. Now, as $\kappa \to +\infty$, we have,

$$\mathcal{B} = \widehat{\sigma}\,\kappa^3 \to +\infty\,.$$

This implies that, as $\kappa \to +\infty$,

$$\mu_1(\kappa, H) \geq \kappa^2\,\widehat{\Lambda}_1\left(B_0, a, \frac{H}{\kappa^2}\right) + \mathcal{O}(\kappa^{\frac{11}{6}})\,.$$

This finish the proof of the theorem. $\square$

4.11.2 Upper bound

The next theorem is a generalization of the results in [42] and [38] valid when the pinning term $a(\kappa, x) = a(x)$ is independent of κ and non-constant.

We denote by $\mu_{\mathcal{B},\widehat{\sigma}}$ the lowest eigenvalue of the operator $P^{\Omega}_{\mathcal{B}\mathbf{F},-\left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}}a}$ i.e.

$$\mu_{\mathcal{B},\widehat{\sigma}} = \inf_{\psi\in H^1(\Omega)\setminus\{0\}} \frac{\mathcal{Q}^{\Omega}_{\mathcal{B}\mathbf{F},-\left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}}a}(\psi)}{\|\psi\|^2_{L^2(\Omega)}}\,.$$

Proposition 4.11.3. *Suppose that $\Gamma \neq \varnothing$ and $\widehat{\lambda}_{\max} > 0$. There exist positive constants C and B_0 such that, for $\widehat{\sigma} \in (0, \widehat{\lambda}_{\max}]$, $a \in C^1(\overline{\Omega})$ and $\mathcal{B} \geq \mathcal{B}_0$, we have,*

$$\mu_{\mathcal{B},\widehat{\sigma}} \leq \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}} \left(\widehat{\Lambda}_1(B_0, a, \widehat{\sigma}) - C\mathcal{B}^{-\frac{1}{18}}\right). \tag{4.11.16}$$

Proof. Let $x_0 \in \Gamma$. In [42, 38], a quasi-mode $u(\mathcal{B}, x_0; x)$ is constructed such that, $\operatorname{supp} u(\mathcal{B}, x_0; \cdot) \subset \overline{\Omega} \cap B(0, \mathcal{B}^{-1/18})$ and,

$$\forall\ \mathcal{B} \geq \mathcal{B}_0\,, \quad \frac{\displaystyle\int_{\Omega} |(\nabla - i\mathcal{B}\mathbf{F})u(\mathcal{B}, x_0; x)|^2\, dx}{\displaystyle\int_{\Omega} |u(\mathcal{B}, x_0; x)|^2\, dx} \leq \mathcal{B}^{\frac{2}{3}}\left(\Lambda(x_0) + C\mathcal{B}^{-1/18}\right),$$

where $\mathcal{B}_0$ and C are constants independent of the point x_0 and the parameter $\mathcal{B}$, and

$$\Lambda(x_0) = \begin{cases} \lambda_0\, |\nabla B_0(x_0)|^{\frac{2}{3}} & \text{if } x_0 \in \Gamma \cap \Omega\,, \\ \lambda(\mathbb{R}^2_+, \theta(x_0))\, |\nabla B_0(x_0)|^{\frac{2}{3}} & \text{if } x_0 \in \Gamma \cap \partial\Omega\,. \end{cases}$$

Using the smoothness of the function $a(\cdot)$, we get in the support of $u(\mathcal{B}, x_0; \cdot)$,

$$|a(x) - a(x_0)| \leq C\mathcal{B}^{-1/18}\,.$$

Thus, we deduce that,

$$\frac{\mathcal{Q}^{\Omega}_{\mathcal{B}\mathbf{F},-\left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}}a}(u(\mathcal{B}, x_0; \cdot)}{\|u(\mathcal{B}, x_0; \cdot)\|^2_{L^2(\Omega)}} \leq \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}} \left(\widehat{\sigma}^{\frac{2}{3}}\Lambda(x_0) - a(x_0) + C\mathcal{B}^{-1/18}\right).$$

Thanks to the min-max principle, we deduce that,

$$\mu_{\mathcal{B},\widehat{\sigma}} \leq \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}} \left(\widehat{\sigma}^{\frac{2}{3}}\Lambda(x_0) - a(x_0) + C\mathcal{B}^{-1/18}\right).$$

Since this is true for all $x_0 \in \Gamma$, we deduce that,

$$\mu_{\mathcal{B},\widehat{\sigma}} \le \left(\frac{\mathcal{B}}{\widehat{\sigma}}\right)^{\frac{2}{3}} \left(\widehat{\Lambda}_1(B_0, a, \widehat{\sigma}) + C\mathcal{B}^{-1/18}\right),$$

where $\widehat{\Lambda}_1(B_0, a, \widehat{\sigma})$ is introduced in (4.11.2). □

Proposition 4.11.3 permits to obtain :

Theorem 4.11.4. *Let $\widehat{\lambda}_{\max} > 0$. Suppose that $\Gamma \neq \varnothing$ and $a \in C^1(\overline{\Omega})$. There exist two constants $C_1 > 0$ and $\kappa_0 > 0$ such that, if,*

$$\kappa \ge \kappa_0\,, \quad \text{and} \quad \kappa_0\kappa^{-1} < H < \widehat{\lambda}_{\max}\kappa^2 \tag{4.11.17}$$

then

$$\mu_1(\kappa, H) \le \kappa^2\, \widehat{\Lambda}_1\left(B_0, a, \frac{H}{\kappa^2}\right) + C_1\kappa^{\frac{11}{6}}\,, \qquad \text{as}\, \kappa \to +\infty\,.$$

Proof. To apply the results of Proposition 4.11.3, we take $\mathcal{B} = \kappa H$ and $\widehat{\sigma} = \frac{H}{\kappa^2}$. We see for κ sufficiently large that $\widehat{\sigma} \in (0, \widehat{\lambda}_{\max})$ and $\mathcal{B}$ large. □

Theorem 4.11.4 is valid when $\kappa H \ge \kappa_0$ and κ_0 is sufficiently large.

4.12 Proof of Theorem 4.1.7

4.12.1 Analysis of $H^{loc}_{C_3}$.

In this subsection we will prove Theorem 4.1.7 for $\underline{H}^{loc}_{C_3}$ and $\overline{H}^{loc}_{C_3}$. We first recall some useful results from [42] about the relation between the eigenvalues λ_0 and $\lambda(\mathbb{R}^2_+, \theta)$, introduced in (4.1.31) and in (4.1.33).

Theorem 4.12.1.

(i) $\lambda(\mathbb{R}^2_+, 0) = \lambda_0$.

(ii) If $0 < \theta < \pi$, then $\lambda(\mathbb{R}^2_+, \theta) < \lambda_0$.

The next proposition gives the region where $\mu_1(\kappa, H) < 0$ that allows us to obtain an information about $\underline{H}^{loc}_{C_3}$ (see (4.1.29)) in the case when the magnetic field B_0 is constant with a pining term.

Proposition 4.12.2. *Suppose that $\{a > 0\} \neq \varnothing$ and $\Gamma \neq \varnothing$. There exist constants $C > 0$ and $\kappa_0 \ge 0$ such that if*

$$\kappa \ge \kappa_0\,, \qquad H \le \max\left(\sup_{x\in\Gamma\cap\Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|}, \sup_{x\in\Gamma\cap\partial\Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+, \theta(x))^{\frac{3}{2}}|\nabla B_0(x)|}\right)\kappa^2 - C\,\kappa^{\frac{11}{6}}\,, \tag{4.12.1}$$

then,

$$\mu_1(\kappa, H) < 0\,.$$

Moreover,

$$\max\left(\sup_{x\in\Gamma\cap\Omega}\frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|},\ \sup_{x\in\Gamma\cap\partial\Omega}\frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+,\theta(x))^{\frac{3}{2}}|\nabla B_0(x)|}\right)\kappa^2 - C\,\kappa^{\frac{11}{6}} \leq \underline{H}^{loc}_{C_3}\,.$$

Proof. We have two cases :
Case 1. Here, we suppose that,

$$\sup_{x\in\Gamma\cap\overline{\Omega}}\frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|} > \sup_{x\in\Gamma\cap\partial\Omega}\frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+,\theta(x))^{\frac{3}{2}}|\nabla B_0(x)|}\,.$$

Thanks to the assumption in (4.1.4), we have, for all $x\in\Gamma\cap\partial\Omega$, $0<\theta(x)<\pi$. Theorem 4.12.1 then tells us that,

$$\forall\, x\in\Gamma\cap\partial\Omega\,,\quad \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+,\theta(x))^{\frac{3}{2}}|\nabla B_0(x)|} > \frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|}\,.$$

Thus, there exists $x_0\in\Omega\cap\Gamma$ such that (the supremum of $\frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|}$ in $\Gamma\cap\overline{\Omega}$ can not be attained on the boundary),

$$\sup_{x\in\Gamma\cap\overline{\Omega}}\frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|} = \frac{a(x_0)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x_0)|}\,.$$

If (4.12.1) is satisfied for some $C>0$, then,

$$\frac{H}{\kappa^2} \leq \frac{a(x_0)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x_0)|} - C\,\kappa^{-\frac{1}{6}}\,,$$

that we can write in the form,

$$\kappa^2\left(\lambda_0\left(\frac{H}{\kappa^2}|\nabla B_0(x_0)|\right)^{\frac{2}{3}} - a(x_0)\right) \leq -C\,M\,\kappa^{\frac{11}{6}}\,, \tag{4.12.2}$$

where $M>0$ is a constant independent of C.

Suppose that $\kappa H \geq \mathcal{B}_0$ where $\mathcal{B}_0$ is selected sufficiently large such that we can apply Theorem 4.11.4. (Thanks to Lemma 4.9.8, $\mu_1(\kappa,H)<0$ when $\kappa H<\mathcal{B}_0$).

By Theorem 4.11.4, there exist positive constants C_1 and κ_0 such that, for $\kappa \geq \kappa_0$,

$$\begin{aligned}\mu_1(\kappa, H) &\leq \kappa^2 \inf_{x \in \Gamma \cap \overline{\Omega}} \left(\lambda_0 \left(\frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{\frac{2}{3}} - a(x) \right) + C_1 \kappa^{\frac{11}{6}} \\ &\leq \kappa^2 \left(\lambda_0 \left(\frac{H}{\kappa^2} |\nabla B_0(x_0)| \right)^{\frac{2}{3}} - a(x_0) \right) + C_1 \kappa^{\frac{11}{6}} \\ &\leq (C_1 - C\, M) \kappa^{\frac{11}{6}} .\end{aligned} \tag{4.12.3}$$

By choosing C such that $C\, M > C_1$, we get,

$$\mu_1(\kappa, H) < 0 .$$

Case 2. Here, we suppose that

$$\sup_{x \in \Gamma \cap \partial\Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+, \theta(x))^{\frac{3}{2}} |\nabla B_0(x)|} \geq \sup_{x \in \Gamma \cap \overline{\Omega}} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}} |\nabla B_0(x)|} .$$

The assumption in (4.12.1) and the upper bound in Theorem 4.11.4 give us, for all $\kappa \geq \kappa_0$, $\kappa H \geq \mathcal{B}_0$ and $\mathcal{B}_0$ a sufficiently large constant,

$$\mu_1(\kappa, H) \leq (C_1 - C\, \widetilde{M}) \kappa^{\frac{11}{6}} .$$

where $\widetilde{M} > 0$ is a constant independent of C. By choosing C such that $C\, \widetilde{M} > C_1$, we get,

$$\mu_1(\kappa, H) < 0 .$$

This finishes the proof of the proposition. $\square$

The next proposition gives us a lower bound of $\overline{H}^{loc}_{C_3}$ (see (4.1.29)). This is obtained by localizing the region where $\mu_1(\kappa, H) > 0$ holds.

Proposition 4.12.3. *Suppose that $\{a > 0\} \neq \varnothing$, $\widehat{\lambda}_{\max} > 0$ and $\Gamma = \varnothing$. There exist constants $C > 0$ and $\kappa_0 > 0$ such that if*

$$\begin{aligned}\kappa \geq \kappa_0 , \qquad \widehat{\lambda}_{\max} \kappa &\geq H \\ &> \max \left(\sup_{x \in \Gamma \cap \overline{\Omega}} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}} |\nabla B_0(x)|}, \sup_{x \in \Gamma \cap \partial\Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+, \theta(x))^{\frac{3}{2}} |\nabla B_0(x)|} \right) \kappa^2 + C \kappa^{\frac{11}{6}} ,\end{aligned} \tag{4.12.4}$$

then,

$$\mu_1(\kappa, H) > 0 .$$

Moreover,

$$\overline{H}_{C_3}^{loc} \leq \max\left(\sup_{x\in\Gamma\cap\overline{\Omega}} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|}, \sup_{x\in\Gamma\cap\partial\Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+,\theta(x))^{\frac{3}{2}}|\nabla B_0(x)|} \right) \kappa^2 + C\,\kappa^{\frac{11}{6}}\,.$$

Proof. Having in mind the definition of $\widehat{\Lambda}_1$ in (4.11.2), under the assumption in (4.12.4), we get for κ large enough,

$$\widehat{\Lambda}_1\left(B_0, a, \frac{H}{\kappa^2}\right) \geq C\,M\,\kappa^{-\frac{1}{6}}\,, \tag{4.12.5}$$

where $M > 0$ is a constant independent of the constant C.

Thanks to Theorem 4.11.2, we get the existence of positive constants C' and κ_0 such that, for $\kappa \geq \kappa_0$,

$$\mu_1(\kappa, H) \geq (C\,M - C')\,\kappa^{\frac{11}{6}}$$

To finish the proof, we choose C sufficiently large such that $C\,M > C'$. □

4.12.2 Analysis of $H_{C_3}^{cp}$.

Proposition 4.12.4 below is an adaptation of an analogous result obtained in [27] for the functional in (4.1.1) with a constant pinning term. Proposition 4.12.4 is valid when $\Gamma \neq \emptyset$. Proposition 4.12.4 says that, if $(\psi, \mathbf{A})$ is a critical point of the functional in (4.1.1) and H is of order κ^2, then $|\psi|$ is concentrated near the set Γ.

Proposition 4.12.4. *Let $\varepsilon > 0$. There exist two positive constants C and κ_0 such that, if*

$$\kappa \geq \kappa_0\,, \quad H \geq \varepsilon\,\kappa^2\,, \tag{4.12.6}$$

and $(\psi, \mathbf{A})$ is a solution of (4.1.12)*, then*

$$\|\psi\|^2_{L^2(\Omega)} \leq C\,\kappa^{-\frac{1}{4}}\|\psi\|^2_{L^4(\Omega)}\,. \tag{4.12.7}$$

Proof. Let $\lambda = \kappa^{-\frac{1}{2}}$ and $\Omega_\lambda = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \lambda \ \&\ \mathrm{dist}(x, \Gamma) > \lambda\}$. We introduce a function $h \in C_c^\infty(\Omega)$ satisfying

$$0 \leq h \leq 1 \text{ in } \Omega\,, \quad h = 1 \text{ in } \Omega_\lambda\,, \quad \mathrm{supp}\,h \subset \Omega_{\lambda/2}\,,$$

and

$$|\nabla h| \leq \frac{C}{\lambda} \quad \text{in } \Omega\,,$$

where C is a positive constant.
Using (4.8.2), we can prove that (see the detailed proof in [27, Eq. (6.6)] when a is constant),

$$\kappa\,H\int_\Omega |B_0(x)|\,|h\psi|^2\,dx - c\,\kappa\,\|\psi\|_{L^2(\Omega)}\|h\psi\|^2_{L^4(\Omega)} \leq \int_\Omega |(\nabla - i\kappa H\mathbf{A})h\psi|^2\,dx\,.$$

Now, the Cauchy-Schwarz inequality yields,

$$c\,\kappa\,\|\psi\|_{L^2(\Omega)}\|h\psi\|^2_{L^4(\Omega)} \leq c^2\|\psi\|^2_{L^2(\Omega)} + \kappa^2\|h\psi\|^4_{L^4(\Omega)}\,,$$

which implies that

$$\begin{aligned}\int_\Omega \big(\kappa\, H\,|B_0(x)| - \kappa^2\,a(x)\big)\,|h\psi|^2\,dx \leq \int_\Omega |(\nabla - i\kappa H\mathbf{A})h\psi|^2\,dx - \kappa^2\int_\Omega a(x)\,|h\psi|^2\,dx \\ + c^2\|\psi\|^2_{L^2(\Omega)} + \kappa^2\|h\psi\|^4_{L^4(\Omega)}\,.\end{aligned}$$

We may use a localization formula as the one in (4.10.14) (but with $\chi_\kappa = h$) to write,

$$\begin{aligned}\int_\Omega \big(\kappa\, H\,|B_0(x)| - \kappa^2\,a(x)\big)\,|h\psi|^2\,dx &\leq c^2\int_\Omega |\psi|^2\,dx + \int_\Omega |\nabla h|^2|\psi|^2\,dx + \kappa^2\int_\Omega (h^4 - h^2)|\psi|^4\,dx \\ &\leq c^2\int_\Omega |\psi|^2\,dx + \int_\Omega |\nabla h|^2|\psi|^2\,dx\,.\end{aligned}$$

Here, we have used the fact that $h^4 \leq h^2$ since $0 \leq h \leq 1$.

By assumption (4.1.4), $|\nabla B_0|$ does not vanish on Γ, hence

$$|B_0(x)| \geq \frac{1}{M}\,\kappa^{-\frac12} \qquad \text{in} \quad \{\mathrm{dist}(x,\Gamma) \geq \lambda\}\,, \tag{4.12.8}$$

for some constant $M > 0$.
Thus, by using (4.1.10) and (4.12.6), we get,

$$\left(\frac{\varepsilon}{M}\,\kappa^{\frac52} - \kappa^2\,\overline{a}\right)\int_\Omega |h\psi|^2\,dx \leq c^2\int_\Omega |\psi|^2\,dx + \int_\Omega |\nabla h|^2|\psi|^2\,dx\,.$$

Writing $\int_\Omega |\psi|^2\,dx = \int_\Omega |h\psi|^2\,dx + \int_\Omega (1-h^2)|\psi|^2\,dx$ and using the assumption on h, we have,

$$\left(\frac{\varepsilon}{M}\,\kappa^{\frac52} - \kappa^2\,\overline{a} - c^2\right)\int_\Omega |h\psi(x)|^2\,dx \leq (c^2 + C\,\kappa)\int_{\Omega\setminus\Omega_\lambda} |\psi|^2\,dx\,.$$

For κ large enough, $\frac{\varepsilon}{M}\,\kappa^{\frac52} - \kappa^2\,\overline{a} - c^2 \geq \frac{\varepsilon}{2M}\,\kappa^{\frac52}$ and

$$\int_\Omega |h\psi(x)|^2\,dx \leq 2\frac{M}{\varepsilon}C\,\kappa^{-\frac32}\int_{\Omega\setminus\Omega_\lambda} |\psi|^2\,dx\,.$$

Thanks to the assumption on the support of h, we get further,

$$\int_\Omega |\psi(x)|^2\,dx \leq \left(2\frac{M}{\varepsilon}C\,\kappa^{-\frac32} + 1\right)\int_{\Omega\setminus\Omega_\lambda} |\psi|^2\,dx\,.$$

Recall that $\lambda=\kappa^{-\frac{1}{2}}$. The Cauchy Schwarz inequality yields,

$$\int_{\Omega\setminus\Omega_\lambda}|\psi(x)|^2\,dx\leq|\Omega\setminus\Omega_\lambda|^{1/2}\left(\int_{\Omega\setminus\Omega_\lambda}|\psi|^4\,dx\right)^{\frac{1}{2}}\leq C\,\kappa^{-\frac{1}{4}}\left(\int_{\Omega}|\psi|^4\,dx\right)^{\frac{1}{2}}.$$

This finishes the proof of the proposition. □

Now, we can give an upper bound of the critical field $\overline{H}^{cp}_{C_3}$ in the case when $\Gamma\neq\varnothing$ and with a pining term.

Theorem 4.12.5. *Supose that $\Gamma\neq\varnothing$ and $\{a>0\}\neq\varnothing$. There exists $C>0$ and κ_0 such that, if H satisfies*

$$H>\max\left(\sup_{x\in\Gamma\cap\overline{\Omega}}\frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|},\ \sup_{x\in\Gamma\cap\partial\Omega}\frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+,\theta(x))^{\frac{3}{2}}|\nabla B_0(x)|}\right)\kappa^2+C\,\kappa^{\frac{11}{6}}\,, \tag{4.12.9}$$

then $(0,\mathbf{F})$ is the unique solution to (4.1.12).
Moreover,

$$\overline{H}^{cp}_{C_3}\leq\max\left(\sup_{x\in\Gamma\cap\overline{\Omega}}\frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|},\ \sup_{x\in\Gamma\cap\partial\Omega}\frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+,\theta(x))^{\frac{3}{2}}|\nabla B_0(x)|}\right)\kappa^2+C\,\kappa^{\frac{11}{6}}\,.$$

Proof. In light of the result in Theorem 4.8.5, we may assume the extra condition that $H\leq\lambda_{\max}\kappa^2$ for a sufficiently large constant $\lambda_{\max}$.

We take the constant C in (4.12.9) as in Proposition 4.12.3. In that way, under the assumption in (4.12.9), we have

$$\mu_1(\kappa,H)<0\,. \tag{4.12.10}$$

Suppose now that $(\psi,\mathbf{A})$ is a solution of (4.1.12) with $\psi\neq0$. Similarly, as in the proof of Theorem 4.10.4, we have,

$$-\top\geq\mu_1(\kappa,H)\,\|\psi\|^2_{L^2(\Omega)}-C\sqrt{\top}\,\kappa\,\|\psi\|^2_{L^2(\Omega)}\,, \tag{4.12.11}$$

where $\top=\kappa^2\|\psi\|^4_{L^4(\Omega)}$ is introduced in (4.10.17).

To apply the result of Proposition 4.12.4, we take

$$\varepsilon=\frac{1}{2}\max\left(\sup_{x\in\Gamma\cap\overline{\Omega}}\frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|},\ \sup_{x\in\Gamma\cap\partial\Omega}\frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+,\theta(x))^{\frac{3}{2}}|\nabla B_0(x)|}\right),$$

and get,

$$\|\psi\|^2_{L^2(\Omega)}\leq C\,\kappa^{-\frac{1}{4}}\|\psi\|^2_{L^4(\Omega)}=C\kappa^{-\frac{5}{4}}\sqrt{\top}\,. \tag{4.12.12}$$

Putting (4.12.12) into (4.12.11), we obtain,

$$-\top\geq\mu_1(\kappa,H)\,\|\psi\|^2_{L^2(\Omega)}-C'\,\kappa^{-\frac{1}{4}}\,\top\,.$$

We conclude that, for $\kappa \geq \kappa_0$ and κ_0 a sufficiently large constant, $\mu_1(\kappa, H) < 0$, which is in contradiction with (4.12.10). Therefore, we conclude that $\psi = 0$. $\square$

Following the argument given in Proposition 4.10.5, we get :

Proposition 4.12.6. *Supose that $\Gamma \neq \varnothing$ and $\{a > 0\} \neq \varnothing$. There exists $C > 0$ and κ_0 such that, if $\kappa \geq \kappa_0$ and H satisfies*

$$H \leq \max\left(\sup_{x\in\Gamma\cap\overline{\Omega}} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|}, \sup_{x\in\Gamma\cap\partial\Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+, \theta(x))^{\frac{3}{2}}|\nabla B_0(x)|} \right) \kappa^2 - C\,\kappa^{\frac{11}{6}}\,, \tag{4.12.13}$$

then there exists a solution $(\psi, \mathbf{A})$ of (4.1.12) with $\|\psi\|_{L^2(\Omega)} \neq 0$.
Moreover,

$$\max\left(\sup_{x\in\Gamma\cap\overline{\Omega}} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|}, \sup_{x\in\Gamma\cap\partial\Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+, \theta(x))^{\frac{3}{2}}|\nabla B_0(x)|} \right) \kappa^2 - C\,\kappa^{\frac{11}{6}} \leq \underline{H}_{C_3}^{cp}\,.$$

End of the proof of Theorem 4.1.7

All the critical fields are contained in the interval $[\underline{H}_{C_3}^{loc}, \overline{H}_{C_3}^{cp}]$ (the proof of this statement is exactly as the one given for (4.10.24) and (4.10.25)).
By Proposition 4.12.2 and Theorem 4.12.5, we get the existence of positive constants C and κ_0, such that for $\kappa \geq \kappa_0$,

$$\begin{aligned}
&\max\left(\sup_{x\in\Gamma\cap\overline{\Omega}} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|}, \sup_{x\in\Gamma\cap\partial\Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+, \theta(x))^{\frac{3}{2}}|\nabla B_0(x)|} \right) \kappa^2 - C\,\kappa^{\frac{11}{6}} \leq \underline{H}_{C_3}^{loc} \leq \overline{H}_{C_3}^{cp} \\
&\qquad \leq \max\left(\sup_{x\in\Gamma\cap\overline{\Omega}} \frac{a(x)^{\frac{3}{2}}}{\lambda_0^{\frac{3}{2}}|\nabla B_0(x)|}, \sup_{x\in\Gamma\cap\partial\Omega} \frac{a(x)^{\frac{3}{2}}}{\lambda(\mathbb{R}^2_+, \theta(x))^{\frac{3}{2}}|\nabla B_0(x)|} \right) \kappa^2 + C\,\kappa^{\frac{11}{6}}\,.
\end{aligned} \tag{4.12.14}$$

As a consequence, we have proved that the asymptotics in Theorem 4.1.7 is valid for for the six critical fields in (4.1.27), (4.1.28) and (4.1.29).

Annexe A

A Theorem à la Sandier-Serfaty

In this chapter, we present a detailed proof of Theorem A.1.1 that will imply the conclusion in Proposition 3.7.3. Theorem A.1.1 and its proof are given by Sandier-Serfaty in [46].

A.1 Statement of the Theorem

Let us consider two positive parameters $h_{\rm ex}$ and ε. If $K \subset \mathbb{R}^2$ is an open set and $(u, A) \in H^1(K, \mathbb{C}) \times H^1(K, \mathbb{R}^2)$, then we define the energy $J_K(u, A)$ as follows,

$$J_K(u, A) = \frac{1}{2}\int_\Omega \left(|(\nabla - iA)u|^2 + |h - h_{\rm ex}|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right) dx\,. \tag{A.1.1}$$

In this chapter, we will use the notation that if $a(\varepsilon)$ and $b(\varepsilon)$ are two non-negative functions of ε, then $a \ll b$ means that $a(\varepsilon) = \delta(\varepsilon)b(\varepsilon)$ and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0_+$.

The aim of this chapter is to present a detailed proof of

Theorem A.1.1. *Let $f :]0, +\infty[\longrightarrow]0, +\infty[$ be a function satisfying*

$$\lim_{t \longrightarrow 0} f(t) = 0\,. \tag{A.1.2}$$

There exists two positive constants ε_0 and $C > 0$, such that if

$$\varepsilon \le \varepsilon_0\,, \qquad |\log \varepsilon| \ll h_{\rm ex} \ll \frac{1}{\varepsilon^2}\,, \tag{A.1.3}$$

$$L(\varepsilon) \ll h_{\rm ex}\ell^2 \ll \min(h_{\rm ex}, L(\varepsilon)^2) \qquad \text{with} \qquad L(\varepsilon) = \log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\,, \tag{A.1.4}$$

$$K \subset \mathbb{R}^2 \text{ is a square of side-length } \ell\,,$$

and $(u, A) \in C^1(K, \mathbb{C}) \times C^1(\Omega, \mathbb{R}^2)$ satisfies,

$$J_K(u, A) \le \frac{1}{2} h_{\rm ex}\, \ell^2 L(\varepsilon) \left(1 + f(\varepsilon)^{\frac{1}{2}}\right), \quad \text{as } \varepsilon \to 0\,, \tag{A.1.5}$$

then, there exist disjoint disks $B_1, ..., B_k$ with the following properties :

- *The sum of the radii of the disks $B_1, ..., B_k$ is less than $h_{\rm ex}^{-\frac{1}{2}}$;*
- $|u| > \frac{1}{2}$ *on* ∂B_i *;*
- *If d_{B_i} is the winding number of $\frac{u}{|u|}$ restricted to ∂B_i then,*

$$2\pi \sum_{i=1}^{k} |d_{B_i}| \leq h_{\rm ex}\ell^2 \left(1 + C\max\left(f(\varepsilon)^{1/2}, L(\varepsilon)^{-1}, \frac{\log L(\varepsilon)}{L(\varepsilon)}\right)\right) ,$$

and

$$2\pi \sum_{i=1}^{k} d_{B_i} = h_{\rm ex}\ell^2 \left(1 + \left(\frac{L(\varepsilon)}{h_{\rm ex}\,\ell^2}\right)^{\frac{1}{3}}\right) .$$

Notice that, in Theorem A.1.1, the configuration (u, A) is not a critical point of the functional J_K. Furthermore, under the assumptions on ε and $h_{\rm ex}$, it is clear that,

$$\lim_{\varepsilon\to 0_+} L(\varepsilon) = \infty\,, \quad \lim_{\varepsilon\to 0_+} \left(\frac{L(\varepsilon)}{h_{\rm ex}\,\ell^2}\right)^{\frac{1}{3}} \quad \text{and} \quad \lim_{\varepsilon\to 0_+} \frac{\log L(\varepsilon)}{L(\varepsilon)} = 0\,.$$

The proof of Theorem A.1.1 will occupy the rest of this chapter.

A.2 Preliminaries

In this section, we collect some notions and theorems from [44, Chapter 4].

Notation.

We will use the following notation :

- The letter C denotes a positive constant that is independent of the parameter ε, and whose value may change from a formula to another.
- If B is a ball, $r(B)$ denotes its radius and if $\mathcal{B}$ is a collection of balls, then $r(\mathcal{B})$ is the sum of the radii of the balls in the collection.
- For $\lambda \geq 0$ the ball λB is the ball with same center as B and radius multiplied by λ. If $\mathcal{B}$ is a collection of balls, then $\lambda\mathcal{B} = \{\lambda B : B \in \mathcal{B}\}$
- $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure in $\mathbb{R}^2$.
- Given $\ell > 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$, $Q_\ell(x) = (-\ell/2 + x_1, \ell/2 + x_1) \times (-\ell/2 + x_2, \ell/2 + x_2)$ denotes the square of side length ℓ centered at x and we write $K = Q_\ell(0)$.
- For $t \in \mathbb{R}_+$ and $u \in C^1(K, \mathbb{C})$, we denote by

$$\Omega_t = \{x \in K \mid |u(x)| < t\}\,, \qquad \gamma_t = \partial\Omega_t \subset \{x \in \overline{K} \mid |u(x)| = t\}\,. \tag{A.2.1}$$

Definition A.2.1.

- **[Radius of a compact set]** *:*
  Let $\omega \subset \mathbb{R}^2$ be a compact set. We define the radius of ω,

$$r(\omega) = \inf\{r(B_1) + ... + r(B_k)\}\,,$$

  where the infimum is taken over all finite coverings of ω by closed balls $B_1, ..., B_k$.

- **Degree of $\mathbb{S}^1$ valued functions :**
  Suppose that $\varphi : \Omega \to \mathbb{R}$ is a C^1 function, $\psi = e^{i\varphi}$ and $B \subset \mathbb{R}^2$ is a disk. The degree of ψ in B is defined by

$$d_B = \begin{cases} \deg(e^{i\varphi}, \partial B) = \dfrac{1}{2\pi}\displaystyle\int_{\partial B} \dfrac{\partial \varphi}{\partial \tau} & \text{if } B \subset \Omega \\ 0 & \text{otherwise}\,. \end{cases} \tag{A.2.2}$$

  Here, τ denote the unit tangent vector to $\partial\Omega$ compatible with this orientation.

Remark A.2.2.

1. If $\omega_1 \subset \omega_2$, then $r(\omega_1) \leq r(\omega_2)$.
2. The infimum which defines the radius is not necessarily achieved.
3. The notion of the radius is a subadditive set-function, i.e. $r(\omega_1 \cup \omega_2) \leq r(\omega_1) + r(\omega_2)$.
4. As we shall see in Proposition A.2.6 below, there is a relation ship between $r(\omega)$ and the perimeter of ω.
5. If $\Omega \subset \mathbb{R}^2$ is an open and bounded set, then $r(\overline{\Omega}) = r(\partial\Omega)$.

Lemma A.2.3 (Merging)**.** *Assume B_1 and B_2 are closed balls in $\mathbb{R}^n$ such that $B_1 \cap B_2 \neq \varnothing$. Then there exists a closed ball B such that $r(B) = r(B_1) + r(B_2)$ and $B_1 \cup B_2 \subset B$.*

Remark A.2.4. Lemma A.2.3 is useful in the following way. Suppose that $r(\omega) < A$. By definition of inf and $r(\omega)$, there exists a finite collection of closed balls $(B_k)_{k\in\mathcal{I}}$ such that,

$$\omega \subset \bigcup B_k \quad \text{and} \quad r(\omega) \leq \sum_k r(B_k) \leq A\,.$$

Using Lemma A.2.3, we may construct a new collection of closed balls, $(B_i')_{i\in\mathcal{J}}$ such that,

$$\mathrm{card}(\mathcal{J}) \leq \mathrm{card}(\mathcal{I})\,,$$

and

$$\omega \subset \bigcup B_i' \quad \text{and} \quad r(\omega) \leq \sum_i r(B_i') \leq A\,.$$

To do this, we merge every two intersecting balls in the collection (B_k) into a single ball as in Lemma A.2.3. This process is repeated a finite number of times until one arrives to the collection (B_i').

Theorem A.2.5 (Ball growth). *Let $\mathcal{B}_0$ be a finite collection of disjoint closed balls. There exists a family $\{\mathcal{B}(t)\}_{t\in\mathbb{R}_+}$ of collections of disjoint closed balls such that $\mathcal{B}(0)=\mathcal{B}_0$ and*

1. *For every $s\geq t\geq 0$,*
$$\bigcup_{B\in\mathcal{B}(t)} B\subset \bigcup_{B\in\mathcal{B}(s)} B\,.$$
2. *There exists a finite set $T\subset\mathbb{R}_+$ such that if $[t_0,t_1]\subset\mathbb{R}_+\setminus T$, then $\mathcal{B}(t_1)=e^{t_1-t_0}\mathcal{B}(t_0)$.*
3. *$r(\mathcal{B}(t))=e^t r(\mathcal{B}_0)$ for every $t\in\mathbb{R}_+$.*

Next, we recall a relationship between the notion of the radius and the perimeter :

Proposition A.2.6. *Assume that ω is compact in $\mathbb{R}^2$. It holds,*

$$2\,r(\omega)\leq\mathcal{H}^1(\partial\omega)\,. \tag{A.2.3}$$

Lemma A.2.7. *Assume Ω is an open subset of $\mathbb{R}^2$ and ω a compact subset of Ω. Assume $v:\Omega\backslash\omega\to\mathbb{S}$ is C^1. If $\mathcal{B}$, $\mathcal{B}'$ are two finite collections of disjoint closed balls such that $\omega\subset\cup_{B\in\mathcal{B}}B$ and $\cup_{B\in\mathcal{B}}B\subset\cup_{B'\in\mathcal{B}'}B'$, then*
$$\sum_{B\in\mathcal{B}}|d_B|\geq\sum_{B'\in\mathcal{B}'}|d_{B'}|\,.$$

Here, the notion d_B is introduced in Definition A.2.1.

A.3 Useful inequalities via the co-area formula

In the sequel, K is a square of side-length $\ell\in(0,1)$, $u:\Omega\to\mathbb{C}$, $A:\Omega\to\mathbb{R}^2$ are C^1, and for all $t>0$,
$$\Omega_t=\{x\in K\ :\ |u(x)|>t\}\,,\quad \gamma_t=\partial\Omega_t\subset\{x\in\overline{K}\ :\ |u(x)|=t\}\,.$$

Notice that for all $t>0$, $|u|>0$ in Ω_t. We write $u=|u|e^{i\varphi}$ and define for any $t>0$,

$$\theta(t)=\int_{K\backslash\overline{\Omega}_t}|\nabla\varphi-A|^2\,dx\,. \tag{A.3.1}$$

Clearly θ is a decreasing function, hence almost everywhere differentiable and

$$\theta'(t)\leq 0\quad\text{a.e.} \tag{A.3.2}$$

Lemma A.3.1. *It holds,*
$$J_K(u,A)\geq\int_0^1 a(t)+2tb(t)\,dt\,, \tag{A.3.3}$$
where
$$a(t)=\frac{\sqrt{2}}{2}\,\frac{|1-t^2|}{\varepsilon}\,r(\gamma_t)\,, \tag{A.3.4}$$
and
$$b(t)=\frac{1}{2}\int_{K\backslash\overline{\Omega}_t}|\nabla\varphi-A|^2\,dx+\frac{1}{2}\int_K|h-h_{\rm ex}|^2\,dx\,. \tag{A.3.5}$$

Proof. Knowing that

$$|(\nabla - iA)u|^2 = |\nabla|u||^2 + |u|^2|\nabla\varphi - A|^2\,.$$

We can rewrite $J_K(u,A)$ as follows :

$$J_K(u,A) = \frac{1}{2}\int_K \left[|\nabla|u||^2 + |u|^2|\nabla\varphi - A|^2 + \frac{1}{2\varepsilon^2}|1-|u|^2|^2 + |h - h_{\rm ex}|^2\right]dx\,. \tag{A.3.6}$$

The simple inequality $(a-b)^2 \geq 0$ yields,

$$|\nabla|u||^2 + \frac{1}{2\varepsilon^2}|1-|u|^2|^2 \geq |\nabla|u||\frac{\sqrt{2}|1-|u|^2|}{\varepsilon}\,.$$

Notice that $|u| = t$ in γ_t. Therefore, using the co-area formula, we find that,

$$\int_K |\nabla|u||^2\,dx + \int_K \frac{1}{2\varepsilon^2}|1-|u|^2|^2\,dx \geq \sqrt{2}\int_0^{+\infty}\frac{|1-t^2|}{\varepsilon}\mathcal{H}^1(\gamma_t)\,dt\,.$$

Applying Proposition A.2.6 with $\omega = \overline{\Omega}_t$, we have,

$$\mathcal{H}^1(\gamma_t) \geq 2\,r(\overline{\Omega}_t) \geq 2\,r(\gamma_t)\,.$$

It follows that,

$$\int_K |\nabla|u||^2\,dx + \int_K \frac{1}{2\varepsilon^2}(1-|u|^2)^2\,dx \geq \sqrt{2}\int_0^{+\infty}\frac{|1-t^2|}{\varepsilon}\,r(\gamma_t)\,dt\,. \tag{A.3.7}$$

Next, we write a lower bound of the integral of $|u|^2|\nabla\varphi - A|^2$. We will use the co-area formula two times two write,

$$\begin{aligned}
\int_K |u|^2|\nabla\varphi - A|^2\,dx &= \int_{t\in(0,\infty)} dt\int_{\gamma_t\cap K} t^2\frac{|\nabla\varphi - A|^2}{|\nabla|u||^2}\,dl\\
&= \int_{t\in(0,\infty)} -t^2\frac{d}{dt}\left[\int_t^\ell\left(\int_{\gamma_s\cap K}\frac{|\nabla\varphi - A|^2}{|\nabla|u||^2}\,dl\right)ds\right]dt\\
&= \int_{t\in(0,\infty)} -t^2\frac{d}{dt}\left(\int_{K\setminus\overline{\Omega}_t}|\nabla\varphi - A|^2\,dx\right)dt\\
&= -\int_{t\in(0,\infty)} t^2\theta'(t)\,dt\,,
\end{aligned} \tag{A.3.8}$$

where $\theta(t)$ is introduced in (A.3.1).

Knowing that $(0,1)\subset(0,\infty)$ and $\theta' \leq 0$, we infer from (A.3.8),

$$\int_K |u|^2|\nabla\varphi - A|^2\,dx \geq -\int_0^1 t^2\theta'(t)\,dt\,.$$

Now, an integrating by parts yields,

$$\int_K |u|^2|\nabla\varphi - A|^2\,dx \geq \int_0^1 2t\,\theta(t)\,dt = \int_0^1 2t\left(\int_{K\setminus\overline{\Omega}_t}|\nabla\varphi - A|^2\,dx\right)dt\,. \tag{A.3.9}$$

Putting (A.3.7) and (A.3.9) into (A.3.6), we obtain,

$$J_K(u,A) \geq \frac{\sqrt{2}}{2}\int_0^1 \frac{|1-t^2|}{\varepsilon}\,r(\gamma_t)\,dt + \int_0^1 t\left[\int_{K\setminus\overline{\Omega}_t}|\nabla\varphi - A|^2\,dx + \int_K |h-h_{\rm ex}|^2\,dx\right]dt\,. \tag{A.3.10}$$

This finishes the proof of Lemma A.3.1. □

In the next lemma, B_R denotes the ball of center 0 and radius R.

Lemma A.3.2. *Let $0<r<R<1$, $v: B_R\setminus\overline{B}_r \longrightarrow \mathbb{S}^1$ and $A: B_R\longrightarrow\mathbb{R}^2$ be C^1. Suppose that,*

$$A\in H^1(B_R)\,,\quad v = e^{i\varphi}\quad\text{and}\quad \varphi\in H^1(B_R\setminus\overline{B}_r).$$

It holds,

$$\int_{B_R\setminus\overline{B}_r}|\nabla\varphi - A|^2\,dx + (R-r)\int_{B_r}|h-h_{\rm ex}|^2\,dx \geq 2\pi|d_{B_R}|\left(\log\frac{R}{r} - \frac{R-r}{2} - h_{\rm ex}\frac{R^2-r^2}{2}\right). \tag{A.3.11}$$

Here, $h = \operatorname{curl} A$ and d is the winding number of v restricted to the circle ∂B_R, i.e.

$$d_{B_R} = \deg(e^{i\varphi},\,\partial B_R) = \frac{1}{2\pi}\int_{\partial B_R}\frac{\partial\varphi}{\partial\tau}\,.$$

Remark A.3.3. For all $r\leq t\leq R$, let d_{B_t} be the winding number of v restricted to the circle ∂B_t, i.e.

$$d_{B_t} = \deg(e^{i\varphi},\,\partial B_t) = \frac{1}{2\pi}\int_{\partial B_t}\frac{\partial\varphi}{\partial\tau}\,.$$

Since $|v|=1$ in $B_R\setminus\overline{B}_r$, then, d_{B_t} remains constant, i.e.

$$\forall\ r\leq t\leq R\,,\quad d_{B_t} = d_{B_R}\,.$$

Proof of Lemma A.3.2. Let

$$e(t) = \int_{\partial B_t}\left|\frac{\partial\varphi}{\partial\tau} - A\cdot\tau\right|^2 + \int_{B_t}|h-h_{\rm ex}|^2\,dx\,. \tag{A.3.12}$$

The obvious inequality $|\nabla\varphi - A| \geq |\frac{\partial\varphi}{\partial\tau} - A\cdot\tau|$ and the co-area formula yield, for all $r\leq t\leq R$,

$$\int_{B_R\setminus\overline{B}_r}|\nabla\varphi - A|^2\,dx \geq \int_r^R\left(\int_{\partial B_t}\left|\frac{\partial\varphi}{\partial\tau} - A\cdot\tau\right|^2\right)dt\,.$$

Since, for all $r \leq t \leq R$, $B_t \subseteq B_R$, then we find,

$$E := \int_{B_R\setminus\overline{B}_r} |\nabla\varphi - A|^2\,dx + (R-r)\int_{B_R} |h - h_{\rm ex}|^2\,dx \geq \int_r^R e(t)\,dt\,. \tag{A.3.13}$$

Let $\lambda_t = \displaystyle\int_{B_t} h\,dx$. Stokes formula yields,

$$\int_{B_t} h\,dx = \int_{B_t} \operatorname{curl} A\,dx = \int_{\partial B_t} A\cdot\tau\,.$$

Now, by definition of d_{B_t}, we get,

$$2\pi d_{B_t} = \lambda_t - \int_{\partial B_t}\left(\frac{\partial\varphi}{\partial\tau} - A\cdot\tau\right)\,.$$

Thanks to the Cauchy-Schwarz inequality, we write,

$$\begin{aligned}(2\pi d_{B_t} - \lambda_t)^2 &\leq \left(\int_{\partial B_t}\left|\frac{\partial\varphi}{\partial\tau} - A\cdot\tau\right|\right)^2\\ &\leq \int_{\partial B_t}\left|\frac{\partial\varphi}{\partial\tau} - A\cdot\tau\right|^2 \int_{\partial B_t} dl\\ &\leq 2\pi t\int_{\partial B_t}\left|\frac{\partial\varphi}{\partial\tau} - A\cdot\tau\right|^2,\end{aligned}$$

and

$$(\lambda_t - \pi t^2 h_{\rm ex})^2 \leq \pi t^2\int_{B_t}|h - h_{\rm ex}|^2\,dx\,.$$

Now, returning back to (A.3.12), we can bound $e(t)$ from below as follows,

$$e(t) \geq \frac{1}{2\pi t}(2\pi d_t - \lambda_t)^2 + \frac{1}{\pi t^2}(\lambda_t - \pi t^2 h_{\rm ex})^2\,. \tag{A.3.14}$$

We will minimize the right-hand side of (A.3.14) with respect to λ_t. Define

$$g(x) = \frac{1}{c}(a-x)^2 + \frac{1}{d}(x-b)^2\,,$$

where a, b, c and d are real numbers.

Clearly,

$$g(x) = x^2\left(\frac{1}{c} + \frac{1}{d}\right) - 2x\left(\frac{a}{c} + \frac{b}{d}\right) + \frac{a^2}{c} + \frac{b^2}{d}\,,$$

and it is easy to see that $x_{\min} = \dfrac{\left(\frac{a}{c} + \frac{b}{d}\right)}{\left(\frac{1}{c} + \frac{1}{d}\right)}$ is the minimizer of $g(x)$, i.e.,

$$g(x) \geq g(x_{\min})\,.$$

We compute $g(x_{\min})$ and obtain

$$\begin{aligned}
g(x_{\min}) &= -\frac{\left(\frac{a}{c}+\frac{b}{d}\right)^2}{\left(\frac{1}{c}+\frac{1}{d}\right)} + \frac{a^2}{c} + \frac{b^2}{d} \\
&= -\frac{cd}{c+d}\left(\frac{a^2}{c^2}+\frac{b^2}{d^2}+\frac{2ab}{cd}\right) + \frac{a^2}{c} + \frac{b^2}{d} \\
&= \frac{-2ab}{c+d} + \frac{a^2}{c+d}\left[-\frac{d}{c}+\frac{c+d}{c}\right] + \frac{b^2}{c+d}\left[-\frac{c}{d}+\frac{c+d}{d}\right] \\
&= \frac{(a-b)^2}{c+d}\,.
\end{aligned}$$

This implies that

$$g(x) \geq \frac{(a-b)^2}{c+d}\,. \tag{A.3.15}$$

We select

$$a = 2\pi d_{B_t}\,, \qquad b = \pi t^2 h_{\text{ex}}\,, \qquad c = 2\pi t \qquad \text{and} \qquad d = \pi t^2\,.$$

By this choose, we get from (A.3.15),

$$\begin{aligned}
e(t) &\geq \frac{(\pi t^2 h_{\text{ex}} - 2\pi d_{B_t})^2}{2\pi t + \pi t^2} \\
&\geq \frac{(2\pi d_{B_t})^2 - 4\pi^2 t^2 h_{\text{ex}}|d_{B_t}|}{2\pi t\left(1+\frac{t}{2}\right)}\,.
\end{aligned}$$

Simplifying and using that, for all $0 < t < 1$, $(1+\frac{t}{2})^{-1} \geq (1-\frac{t}{2}) > 0$ and $-(1+\frac{t}{2})^{-1} \geq -1$, we obtain

$$e(t) \geq \frac{2\pi(d_{B_t})^2}{t}\left(1-\frac{t}{2}\right) - 2\pi\, t\, h_{\text{ex}}|d_{B_t}|\,. \tag{A.3.16}$$

Since $d_{B_t} \in \mathbb{Z}$, then $(d_{B_t})^2 \geq |d_{B_t}|$. We insert this into (A.3.16) then we integrating between r and R. In view of (A.3.13) and (A.3.16), we obtain,

$$E \geq 2\pi|d_{B_t}|\left(\log\frac{R}{r} - \frac{R-r}{2} - h_{\text{ex}}\frac{R^2-r^2}{2}\right)\,,$$

which finishes the proof of Lemma A.3.2. □

Define the function $f :]0,+\infty[\times[0,+\infty[\longrightarrow]-\infty,+\infty[$ as follow :

$$f(r,R) = \pi\left(\log\frac{R}{r} - \frac{R-r}{2} - h_{\text{ex}}\frac{R^2-r^2}{2}\right)\,.$$

Lemma A.3.4. *The following properties are true :*

(P1) : $f(r,.)$ *is increasing on* $\left]0, \min\left\{1, \frac{1}{2\sqrt{h_{\text{ex}}}}\right\}\right]$.

(P2) : $f(r,r) = 0$*, and, for any* $r \leq R \leq \min\left\{1, \frac{1}{2\sqrt{h_{\text{ex}}}}\right\}$, $f(r,R) \geq 0$.

(P3) : $f(r,s) + f(s,R) = f(r,R)$.

(P4) : If $(0 \leq r_i \leq R_i)_{1\leq i\leq k}$ *are* $2k$ *positive real numbers,* $(d_i)_{1\leq i\leq k}$ *are integers, and*

$$\forall i,\, j, \qquad \frac{R_i}{r_i} = \frac{R_j}{r_j} = \alpha\,, \qquad \alpha > 1\,,$$

then

$$\sum_{i=1}^{k} |d_i| f(r_i, R_i) \geq \left(\sum_{i=1}^{k} |d_i|\right) f\left(\sum_{i=1}^{k} r_i\,, \sum_{i=1}^{k} R_i\right)\,. \tag{A.3.17}$$

Proof. The proofs of (P1)-(P3) follow by a straightforward computation. We will prove (P4). For all i, $R_i + r_i \geq 0$ and $R_i - r_i \geq 0$. Thus,

$$\begin{aligned}
R_i^2 - r_i^2 &= (R_i + r_i)(R_i - r_i) \\
&\leq \left(\sum_{j=1}^{k} (R_j + r_j)\right)\left(\sum_{j=1}^{k} (R_j - r_j)\right) \\
&= \left(\sum_{j=1}^{k} R_j + \sum_{j=1}^{k} r_j\right)\left(\sum_{j=1}^{k} R_j - \sum_{j=1}^{k} r_j\right) \\
&= \left(\sum_{j=1}^{k} R_j\right)^2 - \left(\sum_{j=1}^{k} r_j\right)^2\,,
\end{aligned}$$

and (we will use that $\alpha = \dfrac{R_i}{r_i} = \dfrac{\sum_{j=1}^{k} R_j}{\sum_{j=1}^{k} r_j}$),

$$\begin{aligned}
f(r_i, R_i) &\geq \pi\left(\log\alpha - \frac{1}{2}\left(\sum_{j=1}^{k} R_j - \sum_{j=1}^{k} r_j\right) - \frac{h_{\mathrm{ex}}}{2}\left(\left(\sum_{j=1}^{k} R_j\right)^2 - \left(\sum_{j=1}^{k} r_j\right)^2\right)\right) \\
&\geq f\left(\sum_{j=1}^{k} r_j, \sum_{j=1}^{k} R_j\right)\,.
\end{aligned}$$

Multiplying the above inequality by $|d_i|$ and summing over i yields the result in (P4). □

A.4 Construction of vortex disks for $\mathbb{S}^1$ valued functions

The next proposition establishes the existence of disjoint disks as in Theorem A.1.1.

Proposition A.4.1. *Let* $\mathcal{V} \subset \mathbb{R}^2$ *be an open set and* $\omega \subset \mathbb{R}^2$ *be a compact set. Assume that* $v : \mathcal{V} \setminus \omega \longrightarrow \mathbb{S}^1$, $A : \mathcal{V} \longrightarrow \mathbb{R}^2$ *be* C^1 *and* $r(\omega) < \frac{1}{2\sqrt{h_{\mathrm{ex}}}}$.

Then, for any $r(\omega) \leq \sigma \leq \frac{1}{2\sqrt{h_{\mathrm{ex}}}}$, *there exists a family* (B_i) *of disjoint disks of radii* r_i *such that*

1. $\sum_i r_i = \sigma$

2. $\omega \subset \cup_i \overline{B}_i$

3. Letting $h = \operatorname{curl} A$ *and* $v = e^{i\varphi}$*, then,*

$$\sum_i \left(\int_{B_i \setminus \omega} |\nabla\varphi - A|^2 \, dx + \int_{B_i} |\operatorname{curl} A - h_{\mathrm{ex}}|^2 \, dx \right) \geq 2\pi \left(\log \frac{\sigma}{r(\omega)} - C \right)_+ \sum_i |d_{B_i}| \,, \quad \text{(A.4.1)}$$

where C *is a positive constant and* d_{B_i} *is the winding number of* v *restricted to* ∂B_i *if* $\overline{B}_i \Subset \mathcal{V}$*, and zero otherwise.*

Proof. By definition of $r(\omega)$ and Remark A.2.4, there exists a finite collection of disjoint closed disks $\mathcal{B}(0) = (B_i(0))_{i\in\mathcal{I}(0)}$ such that,

$$r(\omega) \leq r(\mathcal{B}(0)) \leq \min(2r(\omega), \sigma)\,.$$

Thanks to Theorem A.2.5, for all $t > 0$, there exists a family of disks $\mathcal{B}(t) = (B_i(t))_{i\in\mathcal{I}(t)}$ such that,

$$r(\mathcal{B}(t)) = e^t r(\mathcal{B}(0))\,.$$

Furthermore, there exists a finite set T such that, if $[t_0, t_1] \subset \mathbb{R} \setminus T$, then

$$\mathcal{B}(t_1) = e^{t_1 - t_0} \mathcal{B}(t_0)\,.$$

Let us define $t_* > 0$ by the relation,

$$e^{t_*} r(\mathcal{B}(0)) = \sigma\,.$$

Suppose that $T \cap [0, t_*] = \{t_1, t_2, \cdots, t_n\}$ such that $t_1 < \cdots < t_n$. We will treat the harder case where this set is non-empty and $0 < t_1 < t_n = t_*$. We let $t_0 = 0$.

We have

$$\left(\bigcup_{i\in\mathcal{I}(t_*)} B_i(t_*) \right) \setminus \left(\bigcup_{i\in\mathcal{I}(0)} B_i(0) \right) = \bigcup_{k=1}^n \left\{ \left(\bigcup_{i\in\mathcal{I}(t_k)} B_i(t_k) \right) \setminus \left(\bigcup_{i\in\mathcal{I}(t_{k-1})} B_i(t_{k-1}) \right) \right\}.$$

Let us take

$$0 < \varepsilon < \frac{1}{2} \min_{1\leq k\leq n} (t_k - t_{k-1})\,.$$

Now, we have the inclusion,

$$\bigcup_{k=1}^n \left\{ \left(\bigcup_{i\in\mathcal{I}(t_{k-1})} B_i((1-\varepsilon)t_k) \right) \setminus \left(\bigcup_{i\in\mathcal{I}(t_{k-1})} B_i((1+\varepsilon)t_{k-1}) \right) \right\} \subset \left(\bigcup_{i\in\mathcal{I}(t_*)} B_i(t_*) \right) \setminus \left(\bigcup_{i\in\mathcal{I}(0)} B_i(0) \right).$$

In light of the obvious inclusion,

$$\left(\bigcup_{i\in\mathcal{I}(t_*)} B_i(t_*)\right) \setminus \left(\bigcup_{i\in\mathcal{I}(0)} B_i(0)\right) \subset \left(\bigcup_{i\in\mathcal{I}(t_*)} B_i(t_*)\right) \setminus \omega\,,$$

we get,

$$\begin{aligned}
&\int_{\bigcup B_i(t_*)\setminus\omega} |\nabla\varphi - A|^2\,dx + \int_{\bigcup B_i(t_*)} |\operatorname{curl} A - h_{\rm ex}|^2\,dx \\
&\quad\geq \sum_{k=1}^n \sum_{i\in\mathcal{I}(t_{k-1})} \left(\int_{B_i(t_k)\setminus B_i(t_{k-1})} |\nabla\varphi - A|^2\,dx + \int_{B_i(t_k)} |\operatorname{curl} A - h_{\rm ex}|^2\,dx\right) \\
&\quad\geq \sum_{k=1}^n \sum_{i\in\mathcal{I}(t_{k-1})} \left(\int_{B_i((1-\varepsilon)t_k)\setminus B_i((1+\varepsilon)t_{k-1})} |\nabla\varphi - A|^2\,dx + \int_{B_i((1+\varepsilon)t_{k-1})} |\operatorname{curl} A - h_{\rm ex}|^2\,dx\right).
\end{aligned}$$

By Theorem A.2.5,

$$\mathcal{B}((1-\varepsilon)t_k) = e^{t_k - t_{k-1} - \varepsilon(t_k + t_{k-1})}\mathcal{B}((1-\varepsilon)t_{k-1})\,.$$

We can apply Lemma A.3.2 to write a lower bound for every

$$\int_{B_i((1-\varepsilon)t_k)\setminus B_i((1+\varepsilon)t_{k-1})} |\nabla\varphi - A|^2\,dx + \int_{B_i((1+\varepsilon)t_{k-1})} |\operatorname{curl} A - h_{\rm ex}|^2\,dx\,,$$

then we sum over i and use Lemma A.3.4 to write,

$$\begin{aligned}
&\sum_{i\in\mathcal{I}(t_{k-1})} \left(\int_{B_i((1-\varepsilon)t_k)\setminus B_i((1+\varepsilon)t_{k-1})} |\nabla\varphi - A|^2\,dx + \int_{B_i((1+\varepsilon)t_{k-1})} |\operatorname{curl} A - h_{\rm ex}|^2\,dx\right) \\
&\qquad\qquad \geq 2\left(\sum_{i\in\mathcal{I}(t_{k-1})} |d_{B_i(t_k)}|\right) \times f\big(r(\mathcal{B}((1-\varepsilon)t_k), r(\mathcal{B}((1+\varepsilon)t_{k-1})\big)\,, \qquad \text{(A.4.2)}
\end{aligned}$$

where f is the function introduced in (A.3.17), and for all $t > 0$, $d_{B_i(t_k)}$ is the degree of $v = e^{i\varphi}$ on $\partial B_i(t)$ if $B_i(t) \subset \mathcal{V}$ and $d_{B_i(t_k)} = 0$ otherwise.

Thanks to Lemma A.2.7, we have,

$$\sum_{i\in\mathcal{I}(t_{k-1})} |d_{B_i(t_k)}| \geq \sum_{i\in\mathcal{I}(t_*)} |d_{B_i(t_*)}|\,.$$

We insert this into (A.4.2) then sum over $k \in \{1, \cdots, n\}$. In that way, we get,

$$\begin{aligned}
&\sum_i \left(\int_{B_i(t_*)\setminus\omega} |\nabla\varphi - A|^2\,dx + \int_{B_i(t_*)} |\operatorname{curl} A - h_{\rm ex}|^2\,dx\right) \\
&\qquad\qquad \geq 2\left(\sum_i |d_{B_i(t_*)}|\right) \sum_{k=1}^n f\big(r(\mathcal{B}((1-\varepsilon)t_k), r(\mathcal{B}((1+\varepsilon)t_{k-1})\big)\,. \qquad \text{(A.4.3)}
\end{aligned}$$

Notice that, by Theorem A.2.5, for all $k \in \{1, \cdots, n\}$,

$$r(\mathcal{B}((1-\varepsilon)t_k) = e^{-\varepsilon t_k} r(\mathcal{B}(t_k) \quad \text{and} \quad r(\mathcal{B}((1+\varepsilon)t_{k-1}) = e^{\varepsilon t_{k-1}} r(\mathcal{B}(t_{k-1})\,.$$

Now, since the term on the left hand side of (A.4.3) is independent of $\varepsilon > 0$, we take the limit as $\varepsilon \to 0_+$ and obtain that,

$$\sum_i \left(\int_{B_i(t_*)\setminus\omega} |\nabla\varphi - A|^2\,dx + \int_{B_i(t_*)} |\operatorname{curl} A - h_{\rm ex}|^2\,dx \right) \geq 2\left(\sum_i |d_i(t_*)|\right) \sum_{k=1}^n f\big(r(\mathcal{B}(t_k), r(\mathcal{B}(t_{k-1})\big)\,. \quad \text{(A.4.4)}$$

Now, we use Lemma A.3.4 to write,

$$\sum_{k=1}^n f\big(r(\mathcal{B}(t_k), r(\mathcal{B}(t_{k-1})\big) = f\big(r(\mathcal{B}(t_*), r(\mathcal{B}(0)\big)\,.$$

Note that, by our choice of $r(\mathcal{B}(t_*) = \sigma$ and $r(\mathcal{B}(0) \in [r(\omega), 2r(\omega)]$, and the definition of the f, we infer now from (A.4.4)

$$\sum_i \left(\int_{B_i(t_*)\setminus\omega} |\nabla\varphi - A|^2\,dx + \int_{B_i(t_*)} |\operatorname{curl} A - h_{\rm ex}|^2\,dx \right) \geq 2\pi\left(\sum_i |d_i(t_*)|\right)\left(\log\frac{\sigma}{r(\omega)} - C\right)\,,$$

where C is a constant. If the term $\left(\log\frac{\sigma}{r(\omega)} - C\right) \leq 0$, then we have the obvious estimate,

$$\sum_i \left(\int_{B_i(t_*)\setminus\omega} |\nabla\varphi - A|^2\,dx + \int_{B_i(t_*)} |\operatorname{curl} A - h_{\rm ex}|^2\,dx \right) \geq 0 = \left(\log\frac{\sigma}{r(\omega)} - C\right)_+\,.$$

In that way, we finish the proof of the proposition. □

A.5 Proof of Theorem A.1.1

This section contains the lengthy proof of Theorem A.1.1. We will work under the assumptions in Theorem A.1.1.

We will split the proof into seven steps.

Step 1 : Locating the set of vortices.

Recall that (u, A) is a configuration satisfying (A.1.5). We will prove that there exists $t_\varepsilon \in [1 - L(\varepsilon)^{-1}, 1]$, such that,

$$r(\gamma_{t_\varepsilon}) < \frac{1}{2\sqrt{h_{\rm ex}}}\,, \quad \text{(A.5.1)}$$

where $\gamma_{t_\varepsilon} = \partial\Omega_{t_\varepsilon}$ and $\Omega_{t_\varepsilon} = \{x \in K \ :\ |u(x)| > t_\varepsilon\}$.

Having in mind the definition of a and b in (A.3.4) and (A.3.5), thanks to (A.3.3), we find that,

$$\begin{aligned} J_K(u,A) &\geq \int_0^1 a(t) + 2tb(t)\, dt \\ &\geq \frac{\sqrt{2}}{2} \int_0^1 \frac{|1-t^2|}{\varepsilon}\, r(\gamma_t)\, dt \\ &\geq \frac{\sqrt{2}}{2} \int_{1-L(\varepsilon)^{-1}}^1 \frac{|1-t^2|}{\varepsilon}\, r(\gamma_t)\, dt\,. \end{aligned}$$

Now, the assumption in (A.1.5) ensures that, for $\varepsilon \in (0, \varepsilon_0)$ and ε_0 sufficiently small,

$$h_{\rm ex}\, \ell^2 \log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}} \geq \frac{\sqrt{2}}{2} \int_{1-L(\varepsilon)^{-1}}^1 \frac{|1-t^2|}{\varepsilon}\, r(\gamma_t)\, dt\,.$$

Using the mean value theorem, there exists $t_\varepsilon \in [\frac{1}{2}, 1]$ such that

$$L(\varepsilon)^{-1}\, r(\gamma_{t_\varepsilon}) \frac{|1-t_\varepsilon^2|}{\varepsilon} = \int_{1-L(\varepsilon)^{-1}}^1 \frac{|1-t^2|}{\varepsilon}\, r(\gamma_t)\, dt \leq \frac{2}{\sqrt{2}}\, h_{\rm ex}\, \ell^2 \log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\,.$$

Using that $|1-t^2| \geq \frac{3}{4}$ if $t \in [\frac{1}{2}, 1]$, we get further,

$$r(\gamma_{t_\varepsilon}) \leq \frac{16}{3\sqrt{2}}\, \varepsilon h_{\rm ex}\, \ell^2 \left(\log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\right)^2\,. \tag{A.5.2}$$

With (A.1.3) and (A.1.4), (A.5.2) becomes,

$$\begin{aligned} r(\gamma_{t_\varepsilon}) &\ll \varepsilon \left(\log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\right)^4 \\ &\ll \frac{1}{\sqrt{h_{\rm ex}}}\,. \end{aligned} \tag{A.5.3}$$

Here, we have used the fact that $x \left(\log \frac{1}{x}\right)^4 \longrightarrow 0$ as $x \longrightarrow 0_+$, and $\varepsilon\sqrt{h_{\rm ex}} \longrightarrow 0$ as $\varepsilon \longrightarrow 0_+$. Therefore, for ε small enough, we obtain,

$$r(\gamma_{t_\varepsilon}) < \frac{1}{2\sqrt{h_{\rm ex}}}\,.$$

Step 2 : Construction of vortex disks.

We will apply Proposition A.4.1 with $\mathcal{V} = K$, $v = \frac{u}{|u|}$ and $\omega = \overline{\Omega}_{t_\varepsilon}$.

Thanks to (A.5.1), for ε small enough, we observe that,

$$r(\gamma_{t_\varepsilon}) < \frac{1}{2\sqrt{h_{\rm ex}}}\,.$$

Thus, we can take $\sigma = \frac{1}{2\sqrt{h_{\rm ex}}}$ in Proposition A.4.1. We find a finite collection of closed disks

$(B_i)_{i\in\mathcal{I}}$, such that, every disk B_i has radius r_i,

$$|u| \geq t_\varepsilon > \frac{1}{2} \quad \text{in } K \setminus \cup_i B_i\,, \qquad \sum_i r_i = \frac{1}{2\sqrt{h_{\rm ex}}}\,, \qquad \overline{\Omega}_{t_\varepsilon} \subset \bigcup_i \overline{B}_i\,,$$

and,

$$\frac{1}{2}\sum_i \int_{B_i\setminus\overline{\Omega}_{t_\varepsilon}} |\nabla\varphi - A|^2\,dx + \frac{1}{2}\int_{B_i} |h-h_{\rm ex}|^2\,dx \geq \pi\left(\sum_i |d_{B_i}|\right)\left(\log\frac{1}{r(\gamma_{t_\varepsilon})\sqrt{h_{\rm ex}}} - C\right)_+ . \tag{A.5.4}$$

Step 3 : Upper bound on the total degree.

Since $|u| \geq t_\varepsilon = 1 - L(\varepsilon)^{-1}$ in every $B_i \setminus \overline{\Omega}_{t_\varepsilon}$, then,

$$\frac{1}{2}\sum_i \int_{B_i\setminus\overline{\Omega}_{t_\varepsilon}} |\nabla\varphi - A|^2\,dx + \frac{1}{2}\int_{B_i} |h-h_{\rm ex}|^2\,dx \leq (1+L(\varepsilon)^{-1})J_K(u,A)\,.$$

We insert this into (A.5.4) and use the upper bound in (A.1.5) to obtain,

$$\pi\left(\sum_i |d_{B_i}|\right)\left(\log\frac{1}{r(\gamma_{t_\varepsilon})\sqrt{h_{\rm ex}}} - C\right)_+ \leq \frac{1}{2}h_{\rm ex}\ell^2 L(\varepsilon)(1+2L(\varepsilon)^{-1}+2f(\varepsilon)^{1/2})\,. \tag{A.5.5}$$

We write,

$$\log\frac{1}{r(\gamma_{t_\varepsilon})\sqrt{h_{\rm ex}}} = \log\frac{1}{\varepsilon\sqrt{h_{\rm ex}}} - \log\frac{r(\gamma_{t_\varepsilon})}{\varepsilon}\,.$$

Thanks to the condition in (A.1.4) and the estimate in (A.5.2), we have,

$$\log\frac{r(\gamma_{t_\varepsilon})}{\varepsilon} \leq C\log L(\varepsilon)\,.$$

We insert this into (A.5.5)

$$\pi\left(\sum_i |d_{B_i}|\right)\left(\log\frac{1}{\varepsilon\sqrt{h_{\rm ex}}} - C\log L(\varepsilon) - C\right)_+ \leq \frac{1}{2}h_{\rm ex}\ell^2 L(\varepsilon)(1+2L(\varepsilon)^{-1}+2f(\varepsilon)^{1/2})\,.$$

After simplification, we obtain,

$$2\pi\sum_i |d_{B_i}| \leq h_{\rm ex}\ell^2\left(1 + C\max\left(f(\varepsilon)^{1/2}, L(\varepsilon)^{-1}, \frac{\log L(\varepsilon)}{L(\varepsilon)}\right)\right)\,. \tag{A.5.6}$$

Step 4 : Refinement of the family of disks.

Recall that $u = |u|e^{i\varphi}$ and the definition of the term $b(t)$ in (A.3.5). We know that,

$$b(t) \geq \pi\sum_i |d_{B_i}|\left(\log\frac{1}{r(\gamma_t)\,\sqrt{h_{\rm ex}}} - C\right)_+ , \tag{A.5.7}$$

We will extract from the collection $(B_i)_{i\in\mathcal{I}}$ a sub-collection $(B_i)_{i=1}^k$ such that, for every i, $B_i \Subset K$

and

$$2\pi \sum_{i=1}^{k} d_{B_i} \geq h_{\mathrm{ex}}\ell^2 \left(1 - \frac{C}{t}\Delta\right)_+ \qquad \text{with} \qquad \Delta = \left(\frac{L(\varepsilon)}{h_{\mathrm{ex}}\,\ell^2}\right)^{\frac{1}{3}}. \tag{A.5.8}$$

Notice that, proving (A.5.8) finishes the proof of Theorem A.1.1. We will prove (A.5.8) in four steps.

Step 4.1 : An auxiliary field.

Let $\widehat{A}$ be a minimizer of the following problem :

$$\mathcal{I}(A) = \min_{\substack{A\in H^1(K,\mathbb{R}^2)\\ \mathrm{divA}=0}} \left(\frac{1}{2}\int_{K\setminus\overline{\Omega}_{t_\varepsilon}} |\nabla\varphi - A|^2\,dx + \frac{1}{2}\int_K |\operatorname{curl} A - h_{\mathrm{ex}}|^2\,dx\right). \tag{A.5.9}$$

Starting from a minimizing sequence, it is standard to prove the existence of a minimizer $\widehat{A}$. Let $\hat{h} = \operatorname{curl}\widehat{A}$. We will prove that $\hat{h}$ satisfies,

$$-\nabla^\perp \hat{h} = \nabla\varphi - \widehat{A} \qquad \text{in } K\setminus\overline{\Omega}_{t_\varepsilon} \tag{A.5.10}$$

$$\hat{h} = \mathrm{cst} \qquad \text{in each connected component of } \Omega_{t_\varepsilon} \tag{A.5.11}$$

$$\hat{h} = h_{\mathrm{ex}} \qquad \text{on } \partial K\setminus\Omega_{t_\varepsilon}. \tag{A.5.12}$$

Let $\mathbf{B}\in H^1(K,\mathbb{R}^2)$ and $\mathrm{div}\mathbf{B} = 0$. Therefore, for all s, $\widehat{A}+s\mathbf{B}\in H^1(K,\mathbb{R}^2)$ and $\mathrm{div}(\widehat{\mathrm{A}} + \mathrm{s}\mathbf{B}) = 0$. Let

$$i(s) = \mathcal{I}(\widehat{A} + s\mathbf{B}).$$

Since $\widehat{A}$ is a minimizer of $\mathcal{I}$, then i has a minimum at $s = 0$ and $i'(0) = 0$. By straightforward computations, we find,

$$\begin{aligned} i(s) &= \mathcal{I}(\widehat{A} + s\mathbf{B})\\ &= \frac{1}{2}\int_{K\setminus\overline{\Omega}_{t_\varepsilon}} |\nabla\varphi - (\widehat{A} + s\mathbf{B})|^2\,dx + \frac{1}{2}\int_K |\operatorname{curl}(\widehat{A} + s\mathbf{B}) - h_{\mathrm{ex}}|^2\,dx\,, \end{aligned}$$

and

$$i'(s) = -\int_{K\setminus\overline{\Omega}_{t_\varepsilon}} \mathbf{B}(\nabla\varphi - (\widehat{A} + s\mathbf{B}))\,dx + \int_K \operatorname{curl}\mathbf{B}(\operatorname{curl}(\widehat{A} + s\mathbf{B}) - h_{\mathrm{ex}})\,dx\,.$$

Therefore, $i'(0) = 0$ yields,

$$-\int_{K\setminus\overline{\Omega}_{t_\varepsilon}} \mathbf{B}(\nabla\varphi - \widehat{A})\,dx + \int_K \operatorname{curl}\mathbf{B}(\operatorname{curl}\widehat{A} - h_{\mathrm{ex}})\,dx = 0\,.$$

Integrating by parts the second integral in the l.h.s. yields that $\widehat{h} = \operatorname{curl}\widehat{A}$ is a solution to (A.5.10)-(A.5.12).

Step 4.2 : Estimating the auxiliary field.

Here, we are going to prove that the auxiliary field $\hat{h}$ is close to h. Recall that $|(\nabla - iA)u|^2 = |\nabla|u||^2 + |u|^2|\nabla\varphi - A|^2$, $\widehat{A}$ is a minimizer of the problem (A.5.9) and $|u| > t_\varepsilon = 1 - L(\varepsilon)^{-1}$ in

$K \setminus \overline{\Omega}_{t_\varepsilon}$. Now, (A.5.10) and (A.5.11) yield,

$$\begin{aligned} J_K(u,A) &\geq t_\varepsilon^2 \left(\frac{1}{2} \int_{K\setminus\overline{\Omega}_{t_\varepsilon}} |\nabla\varphi - A|^2\,dx + \frac{1}{2}\int_K |\operatorname{curl} A - h_{\rm ex}|^2\,dx \right) \\ &\geq t_\varepsilon^2 \left(\frac{1}{2} \int_{K\setminus\overline{\Omega_{t_\varepsilon}}} |\nabla\hat{h}|^2\,dx + \frac{1}{2}\int_K |\hat{h} - h_{\rm ex}|^2\,dx \right). \end{aligned}$$

Using (A.1.5), we get,

$$\int_{K\setminus\overline{\Omega_{t_\varepsilon}}} |\nabla\hat{h}|^2\,dx + \int_K |\hat{h} - h_{\rm ex}|^2\,dx \leq \frac{2\,h_{\rm ex}\ell^2}{t_\varepsilon^2} \log\frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\,. \tag{A.5.13}$$

Since $\widehat{h}$ is constant in every connected component of Ω_{t_ε}, then $\nabla\widehat{h} = 0$ in Ω_{t_ε} and consequently, (A.5.13) becomes,

$$\int_K |\nabla\hat{h}|^2\,dx + \int_K |\hat{h} - h_{\rm ex}|^2\,dx \leq \frac{2\,h_{\rm ex}\ell^2}{t_\varepsilon^2} \log\frac{1}{\varepsilon\sqrt{h_{\rm ex}}}\,. \tag{A.5.14}$$

Step 4.3 : The auxiliary field and the degree.

Here, we will explore a relationship between the degree of $u/|u|$ and the auxiliary field $\hat{h}$. Let $(\omega_i)_i$ be the collection of connected components of Ω_{t_ε}. If $\omega_i \Subset K$, let $d_{\omega_i} = \deg(e^{i\varphi}, \partial\omega_i)$.

Now, suppose that $\omega_i \Subset K$. We have, using the definition of the degree, (A.5.10) and Stokes formula,

$$2\pi d_{\omega_i} = \int_{\partial\omega_i} \frac{\partial\varphi}{\partial\tau} = \int_{\partial\omega_i} \widehat{A}\cdot\tau - \int_{\partial\omega_i}\frac{\partial\hat{h}}{\partial n} = \int_{\omega_i}\hat{h} - \int_{\partial\omega_i}\frac{\partial\hat{h}}{\partial n}\,, \tag{A.5.15}$$

where we have essentially used Stoke's theorem and (A.5.10).

Since $\operatorname{curl}(\nabla\varphi) = 0$, then applying the operator curl on (A.5.10) yields,

$$-\Delta\hat{h} = -\operatorname{curl}(\nabla^\perp\hat{h}) = \operatorname{curl}(\nabla\varphi) - \operatorname{curl}\widehat{A} = -\hat{h}\,. \tag{A.5.16}$$

This gives that,

$$-\Delta\hat{h} + \hat{h} = 0\,, \qquad \text{in } K\setminus\overline{\Omega}_{t_\varepsilon}\,. \tag{A.5.17}$$

Let $K' \subset K$ be an open subset of K such that $\partial K' \cap \overline{\Omega}_{t_\varepsilon} = \emptyset$. Suppose that the boundary of K' is piecewise smooth. We integrate (A.5.17) over $K'\setminus\overline{\Omega}_{t_\varepsilon}$. Using Integration by Parts, we obtain,

$$\int_{K'\setminus\overline{\Omega}_{t_\varepsilon}} -\Delta\hat{h} + \hat{h}\,dx = -\int_{\partial K'}\frac{\partial\hat{h}}{\partial n} + \int_{\partial\Omega_t}\frac{\partial\hat{h}}{\partial n} + \int_{K'\setminus\overline{\Omega}_{t_\varepsilon}}\hat{h}\,dx = 0\,.$$

Therefore,

$$-\int_{\partial K'}\frac{\partial\hat{h}}{\partial n} + \int_{K'\setminus\overline{\Omega}_t}\hat{h}\,dx = -\int_{\partial\Omega_{t_\varepsilon}}\frac{\partial\hat{h}}{\partial n} = -\sum_{\{i,\,\omega_i\subset K'\}}\int_{\partial\omega_i}\frac{\partial\hat{h}}{\partial n}\,. \tag{A.5.18}$$

Here, we used the assumption that $\partial K' \cap \overline{\Omega_t} = \emptyset$ to write,

$$\int_{\partial \Omega_t} \frac{\partial \hat{h}}{\partial n} = \sum_{\{i,\, \omega_i \subset K'\}} \int_{\partial \omega_i} \frac{\partial \hat{h}}{\partial n}\,.$$

Using (A.5.18), (A.5.15) and the assumption $\partial K' \cap \overline{\Omega_{t_\varepsilon}} = \emptyset$, we get

$$\begin{aligned}
-\int_{\partial K'} \frac{\partial \hat{h}}{\partial n} + \int_{K'} \hat{h}\,dx &= -\int_{\partial K'} \frac{\partial \hat{h}}{\partial n} + \int_{K' \setminus \overline{\Omega}_{t_\varepsilon}} \hat{h}\,dx + \int_{K' \cap \overline{\Omega}_{t_\varepsilon}} \hat{h}\,dx \\
&= -\sum_{\{i,\, \omega_i \subset K'\}} \int_{\partial \omega_i} \frac{\partial \hat{h}}{\partial n} + \sum_{\{i,\, \omega_i \subset K'\}} \int_{\omega_i} \hat{h} \\
&= \sum_{\{i,\, \omega_i \subset K'\}} 2\pi\, d_{\omega_i}\,. \qquad (A.5.19)
\end{aligned}$$

Step 4.4 : A lower bound of the total degree.

For every $\beta > 0$, we introduce the following subset of the square K,

$$K_\beta = \{x \in K : \operatorname{dist}(x, \partial K) > \beta\}\,.$$

Notice that, for all $0 < \beta < \ell/2$, K_t occupies a square of side-length $\ell - 2\beta$. Later, we will choose $\alpha = \alpha(\varepsilon) \in (0,1)$ such that

$$\frac{1}{\sqrt{h_{\rm ex}}} \ll \alpha \ll \frac{\ell}{4} \quad (\varepsilon \to 0_+)\,. \qquad (A.5.20)$$

Since α satisfies (A.5.20) and $\sum_i r_i = \frac{1}{2\sqrt{h_{\rm ex}}}$, then the (Lebesuge) measure of the set,

$$T = \{\beta \in (0, \alpha] : \partial K_\beta \cap \bigcup_i B_i = \varnothing\}$$

satisfies,

$$|T| \geq \frac{\alpha}{2}\,.$$

To see this, notice that

$$T^c \subset \pi_1\left(\bigcup_i B_i\right) \bigcup \pi_2\left(\bigcup_i B_i\right)\,,$$

where

$$\pi_1 : (x_1, x_2) \mapsto x_1 \quad \text{and} \quad \pi_2 : (x_1, x_2) \mapsto x_2\,.$$

Consequently, we see that,

$$|T^c| \leq \left|\pi_1\left(\bigcup_i B_i\right)\right| + \left|\pi_2\left(\bigcup_i B_i\right)\right| \leq 4\sum_i r_i < \frac{\alpha}{2}\,.$$

Now, consider the set

$$T' = \left\{ \beta \in T : \int_{\partial K_\beta} \left| \frac{\partial \hat{h}}{\partial n} \right|^2 < \frac{8\, h_{\rm ex}\, \ell^2}{\alpha\, {t_\varepsilon}^2} \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}} \right\}.$$

We claim that,

$$|T'| \geq \frac{\alpha}{4}\,.$$

To see this, we write by the co-area formula,

$$\int_{K\setminus K_\alpha} |\nabla \hat{h}|^2\, dx = \int_0^\alpha \left(\int_{\partial K_\beta} |\nabla \hat{h}|^2 \right) d\beta\,.$$

Since $T' \subset T \subset [0,\alpha]$ and $|\nabla \widehat{h}| = |\partial \widehat{h}/\partial n|$, we get further,

$$\int_{K\setminus K_\alpha} |\nabla \hat{h}|^2\, dx \geq \int_{T\setminus T'} \left(\frac{8\, h_{\rm ex}\, \ell^2}{\alpha\, {t_\varepsilon}^2} \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}} \right) d\beta \geq (|T| - |T'|) \frac{8\, h_{\rm ex}\, \ell^2}{\alpha\, {t_\varepsilon}^2} \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}}\,.$$

Now, using (A.5.14), we get,

$$(|T| - |T'|) \frac{8\, h_{\rm ex}\, \ell^2}{\alpha\, {t_\varepsilon}^2} \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}} \leq \frac{2\, h_{\rm ex}\, \ell^2}{{t_\varepsilon}^2} \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}}\,.$$

Since $|T| \geq \frac{\alpha}{2}$, we deduce that $|T'| \geq \frac{\alpha}{4}$.

Now, since $|T'| > 0$, then $T' \neq \emptyset$ and there exists $\beta_0 \in [0,\alpha]$ such that

$$\partial K_{\beta_0} \cap \bigcup_i B_i = \varnothing \qquad \text{and} \qquad \int_{\partial K_{\beta_0}} \left| \frac{\partial \hat{h}}{\partial n} \right|^2 < \frac{8\, h_{\rm ex}\, \ell^2}{\alpha\, {t_\varepsilon}^2} \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}}\,. \tag{A.5.21}$$

In particular, we have that

$$\partial K_{\beta_0} \cap \overline{\Omega_t} = \emptyset\,,$$

and

$$\sum_{B_i \subset K_{\beta_0}} d_{B_i} = \sum_{i,\, \omega_i \subset K_{\beta_0}} d_{\omega_i}\,,$$

where $(\omega_i)_i$ are the connected components of $\overline{\Omega_t}$.

We define the sub-collection of $(B_i)_{i=1}^k$ to be $(B_i)_{B_i \subset K_{\beta_0}})$.

Now, we can apply (A.5.19) with $K' = K_{\beta_0}$ and get,

$$\begin{aligned} 2\pi \sum_{B_i \subset K_{\beta_0}} d_{B_i} &\geq \int_{K_{\beta_0}} \hat{h} - \int_{\partial K_{\beta_0}} \frac{\partial \hat{h}}{\partial n} \\ &\geq \int_{K_{\beta_0}} h_{\rm ex} - \left| \int_{K_{\beta_0}} \hat{h} - h_{\rm ex} \right| - \left| \int_{\partial K_{\beta_0}} \frac{\partial \hat{h}}{\partial n} \right|. \end{aligned} \tag{A.5.22}$$

Since the region K_{β_0} occupies a square of side length $\ell - 2\beta_0$, then,

$$|K_{\beta_0}| = (\ell - 2\beta_0)^2 \qquad \text{and} \qquad |\partial K_{\beta_0}| = 4(\ell - 2\,\beta_0)\,.$$

Since $\beta_0 \in (0, \alpha]$ and $\alpha < \frac{\ell}{4}$, then,

$$(\ell - \alpha)^2 \leq |K_{\beta_0}| \leq \ell^2 \qquad \text{and} \qquad 4(\ell - 2\alpha) \leq |\partial K_{\beta_0}| \leq 4\ell\,.$$

Now, we have the following simple inequalities,

$$\int_{K_{\beta_0}} h_{\mathrm{ex}} \geq h_{\mathrm{ex}}(\ell - \alpha)^2\,,$$

$$\left|\int_{K_{\beta_0}} \hat{h} - h_{\mathrm{ex}}\right| \leq \ell \left(\int_{K_{\beta_0}} |\hat{h} - h_{\mathrm{ex}}|^2\right)^{\frac{1}{2}}$$

and

$$\left|\int_{\partial K_{\beta_0}} \frac{\partial \hat{h}}{\partial n}\right| \leq 2\ell^{\frac{1}{2}} \left(\int_{\partial K_{\beta_0}} \left|\frac{\partial \hat{h}}{\partial n}\right|^2\right)^{\frac{1}{2}}\,.$$

The last two inequalities are obtained via Cauchy-Schwarz. Now, we put these inequalities into (A.5.22), use (A.5.14), (A.5.21) and expand $(\ell - \alpha)^2$ to obtain,

$$\begin{aligned} 2\pi d(t) &\geq h_{\mathrm{ex}}(\ell - \alpha)^2 - \frac{C}{t_\varepsilon}\,\ell^{\frac{3}{2}} \left(\frac{h_{\mathrm{ex}}\,L(\varepsilon)}{\alpha}\right)^{\frac{1}{2}} \\ &\geq \ell^2\,h_{\mathrm{ex}} \left(1 - \frac{4\alpha}{\ell} - \frac{C}{t_\varepsilon}\left(\frac{L(\varepsilon)}{h_{\mathrm{ex}}\,\ell\,\alpha}\right)\right)\,. \end{aligned} \tag{A.5.23}$$

Now, we choose

$$\alpha = \ell \left(\frac{L(\varepsilon)^2}{h_{\mathrm{ex}}\,\ell}\right)^{\frac{1}{3}}\,.$$

Under Assumption (A.1.4), we get, for ε small enough,

$$\alpha \ll \frac{\ell}{4}\,,$$

and,

$$\ell \gg \frac{\sqrt{L(\varepsilon)}}{\sqrt{h_{\mathrm{ex}}}}\,, \qquad \frac{L(\varepsilon)}{h_{\mathrm{ex}}\,\ell^2} \gg \frac{1}{L(\varepsilon)}\,.$$

Thus,

$$\alpha \gg \frac{L(\varepsilon)^{\frac{1}{2}-\frac{1}{3}}}{\sqrt{h_{\mathrm{ex}}}} \gg \frac{1}{\sqrt{h_{\mathrm{ex}}}}\,.$$

Therefore this choice of α respects (A.5.20). Moreover, (A.5.23) becomes,

$$2\pi\,d(t) \geq \ell^2\,h_{\mathrm{ex}} \left(1 - \frac{C}{t_\varepsilon}\Delta\right)\,,$$

with

$$\Delta = \left(\frac{L(\varepsilon)}{h_{\rm ex}\,\ell^2} \right)^{\frac{1}{3}} .$$

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